DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTERS IN SCIENCE-MATHEMATICS

SEMESTER -II

POINT SET TOPOLOGY

DEMATH-2 CORE-2

BLOCK-1

UNIVERSITY OF NORTH BENGAL

Postal Address:

The Registrar,

University of North Bengal,

Raja Rammohunpur,

P.O.-N.B.U.,Dist-Darjeeling,

West Bengal, Pin-734013,

India.

Phone: (O) +91 0353-2776331/2699008

Fax:(0353) 2776313, 2699001

Email: regnbu@sancharnet.in ; regnbu@nbu.ac.in

Wesbsite: www.nbu.ac.in

First Published in 2019

POINT SET TOPOLOGY

POINT SET TOPOLOGY

Introduction to the block-I

Unit 1 Ordered Set : This unit deals with ordered set and Axiom Of Choice.

Unit 2 Well Ordering Set : Deals with well ordering set and Zorn's Lemma.

Unit 3 Ordinal and Cardinal Numbers : This unit deals with Cardinal number and ordinal number.

Unit 4 Topological Space : Deals with topological space and Ordering Topology and Basis of Topology and Closed set, open set topology.

Unit 5 Interior And Boundary Point of a Set : Deals with interior of sets , disjoint sets, Interior Operator and Subspace terminology.

Unit 6 Continuous Mapping : Deals Continuous mapping also Quotient, Open, Perfect Mapping.

Unit 7 Topological Manifold : Deals topological manifold and Enriched Circles**.** Also deals patchwork.

UNIT 1 ORDERED SET

STRUCTURE

1.0 Objective

1.1 Introduction

- 1.1.1 Axiom Of Choice
- 1.1.2 Existence Of Choice Functions
- 1.1.3 Varient
- 1.1.4 Examples
- 1.1.5 Independence

1.2 Ordered Set

- 1.2.1 Duality
- 1.2.2 Extrema

1.3 Partially Ordered Set

- 1.3.1 Extrema
- 1.3.2 Order on the Cartesian Product of Partially Ordered Sets
- 1.3.3 Sum of Partially Ordered Sets
- 1.3.4 Strict and Non-strict Partial Ordered Sets
- 1.3.5 Mapping between Partial Ordered Sets
- 1.3.6 Well Ordered Sets

1.4. Ordinal Numbers

- 1.4.1 Natural Numbers
- 1.4.2 Integers
- 1.4.3 Reals

1.5 Summary

1.6 Keyword

1.7 Exercise

1.8 Answer For Check in Progress

1.9. Suggestion Reading And Reference

1.0 OBJECTIVE

- Learn Partially Ordered Set
- Learn Mapping b/w partial ordered set
- Work on Cartesian Product of partially ordered set
- To Know about Duality
- To know about Extrema

1.1 INTRODUCTION

A totally ordered set is also termed a chain . If the order is partial, so that P has two or more incomparable elements, then the ordered set is a partially ordered set . See Figure 2 for an example. At the other extreme, if no two elements are comparable unless they are equal, then the ordered set is an antichai.

1.1.1 Axiom Of Choice

The axiom of choice, or AC, is an axiom of set theory equivalent to the statement that *the Cartesian product of a collection of non-empty sets is non-empty*. Informally put, the axiom of choice says that given any collection of bins, each containing at least one object, it is possible to make a selection of exactly one object from each bin, even if the collection is infinite. Formally, it states that for every indexed family of nonempty sets there exists an indexed family of elements. The axiom of choice was formulated in 1904 by Ernst Zermelo in order to formalize his proof of the well-ordering theorem.

In many cases, such a selection can be made without invoking the axiom of choice; this is in particular the case if the number of sets is finite, or if a selection rule is available – some distinguishing property that happens to hold for exactly one element in each set. An illustrative example is sets picked from the natural numbers. From such sets, one may always select the smallest number, e.g. in {{4, 5, 6}, {10, 12}, {1, 400, 617, 8000}} the smallest elements are {4, 10, 1}. In this case, "select the smallest number" is a choice function. Even if infinitely many sets were collected from the natural numbers, it will always be possible to choose the smallest element from each set to produce a set. That is, the choice function provides the set of chosen elements. However, no choice function is known for the collection of all non-empty subsets of the real numbers (if there are non-constructible reals). In that case, the axiom of choice must be invoked.

Bertrand Russell coined an analogy: for any (even infinite) collection of pairs of shoes, one can pick out the left shoe from each pair to obtain an appropriate selection; this makes it possible to directly define a choice function. For an *infinite* collection of pairs of socks (assumed to have no distinguishing features), there is no obvious way to make a function that selects one sock from each pair, without invoking the axiom of choice.

Although originally controversial, the axiom of choice is now used without reservation by most mathematicians,[3] and it is included in the standard form of axiomatic set theory, Zermelo–Fraenkel set theory with the axiom of choice (ZFC). One motivation for this use is that a number of generally accepted mathematical results, such as Tychonoff's theorem, require the axiom of choice for their proofs. Contemporary set theorists also study axioms that are not compatible with the axiom of choice, such as the axiom of determinacy. The axiom of choice is avoided in some varieties of constructive mathematics, although there are varieties of constructive mathematics in which the axiom of choice is embraced.

1.1.2 Existence of Choice Function

A choice function is a function f, defined on a collection X of nonempty sets, such that for every set A in X, $f(A)$ is an element of A. With this concept, the axiom can be stated:

Axiom — For any set X of nonempty sets, there exists a choice function f defined on X.

A choice function is defined to exist if there is a "best" (under a binary relation R) element in all non-empty compact subsets of S, the set of all possible alternatives, whereas a demand correspondence exists if there is a "best" element in only the budget sets of S. Some basic restrictions on R are considered. First, if the "at least as good as" sets are closed, then none of the standard restrictions on R are shown to be necessary for the existence of a demand correspondence: the "domination" of finite sets is necessary and sufficient. This is shown to imply that acyclicity of R is necessary and sufficient for the existence of choice functions. Second, if either there is a restriction on convergent P monotone sequences or if R satisfies a regularity condition, then a condition on cyclical sets of alternatives is enough to guarantee the existence of demand correspondences. For the existence of rational choice functions, however, reflexivity, completeness, and transitivity of R, together with the above-mentioned condition on P-monotone sequences, are necessary and sufficient. Finally, if the strictly preferred sets are taken to be convex, then under a restriction weaker than the first, a best element in budget sets exists.

Thus, the negation of the axiom of choice states that there exists a collection of nonempty sets that has no choice function.

Each choice function on a collection *X* of nonempty sets is an element of the Cartesian product of the sets in *X*. This is not the most general situation of a Cartesian product of a family of sets, where a given set can occur more than once as a factor; however, one can focus on elements of such a product that select the same element every time a given set appears as factor, and such elements correspond to an element of the Cartesian product of all *distinct* sets in the family. The axiom of choice asserts the existence of such elements; it is therefore equivalent to:

Given any family of nonempty sets, their Cartesian product is a nonempty set.

1.1.3 Varient

There are many other equivalent statements of the axiom of choice. These are equivalent in the sense that, in the presence of other basic axioms of set theory, they imply the axiom of choice and are implied by it.

One variation avoids the use of choice functions by, in effect, replacing each choice function with its range.

Given any set *X* of pairwise disjoint non-empty sets, there exists at least one set *C* that contains exactly one element in common with each of the sets in *X*.

This guarantees for any partition of a set *X* the existence of a subset *C* of *X* containing exactly one element from each part of the partition.

Another equivalent axiom only considers collections *X* that are essentially powersets of other sets:

For any set A, the power set of A (with the empty set removed) has a choice function.

Authors who use this formulation often speak of the *choice function on A*, but be advised that this is a slightly different notion of choice function. Its domain is the powerset of *A* (with the empty set removed), and so makes sense for any set *A*, whereas with the definition used elsewhere in this article, the domain of a choice function on a *collection of sets* is that collection, and so only makes sense for sets of sets. With this alternate notion of choice function, the axiom of choice can be compactly stated as

Every set has a choice function,

which is equivalent to

For any set A there is a function *f* such that for any non-empty subset B of *A*, *f*(*B*) lies in *B*.

The negation of the axiom can thus be expressed as:

There is a set *A* such that for all functions *f* (on the set of non-empty subsets of *A*), there is a *B* such that *f*(*B*) does not lie in *B*.

1.1.4 Examples

The nature of the individual nonempty sets in the collection may make it possible to avoid the axiom of choice even for certain infinite collections. For example, suppose that each member of the collection *X* is a nonempty subset of the natural numbers. Every such subset has a smallest element, so to specify our choice function we can simply say that it maps each set to the least element of that set. This gives us a definite choice of an element from each set, and makes it unnecessary to apply the axiom of choice.

The difficulty appears when there is no natural choice of elements from each set. If we cannot make explicit choices, how do we know that our set exists? For example, suppose that X is the set of all non-empty subsets of the real numbers. First we might try to proceed as if X were finite. If we try to choose an element from each set, then, because X is infinite, our choice procedure will never come to an end, and consequently, we shall never be able to produce a choice function for all of X. Next we might try specifying the least element from each set. But some subsets of the real numbers do not have least elements. For example, the open interval (0,1) does not have a least element: if x is in (0,1), then so is $x/2$, and $x/2$ is always strictly smaller than x. So this attempt also fails.

Additionally, consider for instance the unit circle S, and the action on S by a group G consisting of all rational rotations. Namely, these are rotations by angles which are rational multiples of π. Here G is countable while S is uncountable. Hence S breaks up into uncountably many orbits under G. Using the axiom of choice, we could pick a single point from each orbit, obtaining an uncountable subset X of S with the property that all of its translates by G are disjoint from X. The set of those translates partitions the circle into a countable collection of disjoint sets, which are all pairwise congruent. Since X is not measurable for any rotation-invariant countably additive finite measure on S, finding an algorithm to select a point in each orbit requires the axiom of choice. See non-measurable set for more details.

The reason that we are able to choose least elements from subsets of the natural numbers is the fact that the natural numbers are well-ordered: every nonempty subset of the natural numbers has a unique least element under the natural ordering. One might say, "Even though the usual ordering of the real numbers does not work, it may be possible to find a different ordering of the real numbers which is a well-ordering. Then our choice function can choose the least element of every set under our unusual ordering." The problem then becomes that of constructing a well-ordering, which turns out to require the axiom of choice for its existence; every set can be well-ordered if and only if the axiom of choice holds.

1.1.5 Independence

Kurt Gödel showed that the *negation* of the axiom of choice is not a theorem of ZF by constructing an inner model (the constructible universe) which satisfies ZFC and thus showing that ZFC is consistent if ZF itself is consistent. In 1963, Paul Cohen employed the technique of forcing, developed for this purpose, to show that, assuming ZF is consistent, the axiom of choice itself is not a theorem of ZF. He did this by constructing a much more complex model which satisfies ZF¬C (ZF with the negation of AC added as axiom) and thus showing that ZF¬C is consistent[15].

Together these results establish that the axiom of choice is logically independent of ZF. The assumption that ZF is consistent is harmless because adding another axiom to an already inconsistent system cannot make the situation worse. Because of independence, the decision whether to use the axiom of choice (or its negation) in a proof cannot be made by appeal to other axioms of set theory. The decision must be made on other grounds.

One argument given in favor of using the axiom of choice is that it is convenient to use it because it allows one to prove some simplifying propositions that otherwise could not be proved. Many theorems which are provable using choice are of an elegant general character: every ideal in a ring is contained in a maximal ideal, every vector space has a basis, and every product of compact spaces is compact. Without the axiom of choice, these theorems may not hold for mathematical objects of large cardinality.

The proof of the independence result also shows that a wide class of mathematical statements, including all statements that can be phrased in the language of Peano arithmetic, are provable in ZF if and only if they are provable in ZFC.[16] Statements in this class include the statement that P = NP, the Riemann hypothesis, and many other unsolved mathematical problems. When one attempts to solve problems in this class, it makes no difference whether ZF or ZFC is employed if the only question is the existence of a proof. It is possible, however, that there is a shorter proof of a theorem from ZFC than from ZF.

The axiom of choice is not the only significant statement which is independent of ZF. For example, the generalized continuum hypothesis (GCH) is not only independent of ZF, but also independent of ZFC. However, ZF plus GCH implies AC, making GCH a strictly stronger claim than AC, even though they are both independent of ZF.

Check in Progress-I

Note: i) Write your answers in the space given below.

Q. 1 Define Independence

Solution

…………………………………………………………………………..

……………………………………………………………………………….

……………………………………………………………………………….

………………………………………………………………………………
…..

Q. 2 State Varient.

Solution

…………………………………………………………………………..

……………………………………………………………………………….

……………………………………………………………………………….

………………………………………………………………………………
…..

1.2 ORDERED SET

Let P be a set and ⊑ be a (partial) order on P. Then P and ⊑ form a *(partially) ordered set*.

If the order is total, so that no two elements of P are incomparable, then the ordered set is a *totally ordered set*. Totally ordered sets are the ones people are first familiar with. See Figure 1 for an example.

A totally ordered set is also termed a *chain*.

If the order is partial, so that P has two or more incomparable elements, then the ordered set is a *partially ordered set*. See Figure 2 for an example.

At the other extreme, if no two elements are comparable unless they are equal, then the ordered set is an *antichain*. See Figure 3.

On any set, = is an order; this is termed the *discrete order* on the set. Any set ordered by = forms an antichain.

It is common for people to refer briefly though inaccurately to an ordered set as an *order*, to a totally ordered set as a *total order*, and to a partially ordered set as a *partial order*. It is usually clear by context whether "order" refers literally to an order (an order relation) or by synecdoche to an ordered set.

Examples:

1. The integers with ≤ form an ordered set (see Figure 1). ≤ is a total order on the integers, so this ordered set is a chain.
2. Any powerset with ⊆ forms an ordered set (see Figure 2). This is a partially ordered set because not all subsets are related by ⊆, for example {a} || {b, r}.
3. A set of unrelated items, ordered by =, is the discrete order on that set and forms an antichain .
4. The classes in java.util with the subclass relation form an ordered set (see Figure 4). This set is partially ordered, because not all classes in the set are related by the subclass relations (for example, Vector and HashSet are not related and are thus incomparable: Vector || HashSet).
5. A set of binary strings with the prefix relation forms an ordered set (see Figure 5). This is set is partially ordered because not all strings are related by the prefix relation, for example 01 || 10.

6. The (non-empty) conjunctions of any of the propositions, q, and r, ordered by implication, form an ordered set . In this set, p∧q implies q, but p∧q neither implies nor is implied by q∧r, so p∧q and q∧r are incomparable (p∧q # q∧r) .

7. The positive integers N with the divisibility relation form an ordered set. The divisibility relation relates m to n if m divides n, written m | n. Thus 2 | 6, and 3 | 6 but not 4 | 6 because 4 does not divide 6. And for any n∈N, 1 | n and n | n. A part of this ordered set is shown in **Figure 7.**

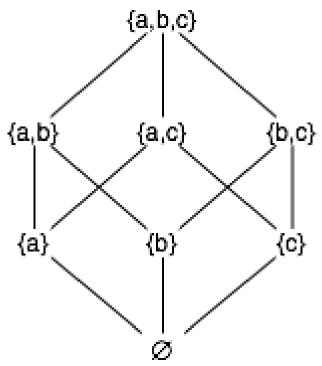


Figure 1.1 The Point of a,b,c in order ⊑

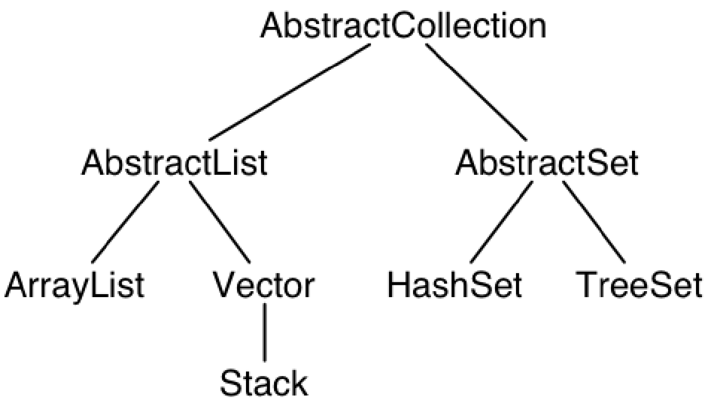


Figure 1.2

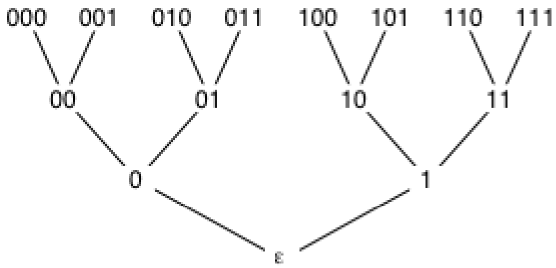


Figure 1.3

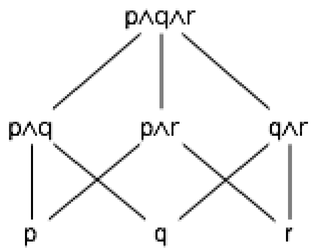


Figure 1.4

1.2.1 Duality

Each ordered set P corresponds to another ordered set P^∂, the *dual* of P, defined by: $y \sqsubseteq x$ in P^∂ iff $x \sqsubseteq y$ in P.

Each statement Φ about P corresponds to a *dual statement* Φ^∂ about P^∂. Φ^∂ is obtained by replacing each occurrence of ⊑ in Φ by ⊒, and each occurrence of ⊒ in Φ by ⊑. Φ is true about P if and only if Φ^∂ is true about P^∂. Generalizing, it can be shown that if a statement Φ is true about all ordered sets, then its dual statement Φ^∂ is also true. This assertion is the *Duality Principle*.

Pairs of dual concepts that are defined in terms of ⊑ and ⊒ (such as upper bound and lower bound, below), are also exchanged in dual statements.

Example: Let Q be the ordered set shown in Figure 7, in which ⊑ is the integer *divides* relation, with the divisor "lower than" the dividend. Then the ordered set of the positive integers to 15 ordered by the converse of *divides* (now with the divisor considered "higher" than the dividend), is the dual Q^∂ of Q. The converse of |, $|^{-1}$, relates two integers if one divides the other, but unlike | it classifies the numerically-smaller integer as the "higher" one by this relation, so that for this order $2 \sqsubseteq 1$, for example. Q^∂ is shown in Figure 8. In Q, $4 \sqsubseteq 8$, so we know without looking at Figure 8 that in Q^∂, the dual statement $4 \sqsupseteq 8$ holds in the relation for that ordered set.

1.2.2 Extrema

Let S be an ordered set.

- $u \in S$ is said to be *maximal* in S iff there is no $v \in S$ such that $u \leq v$. A set may have any number of maximal elements, including zero.

- If u is S's only maximal element, then u is the *maximum* of S.
- The maximum element u (if it exists) is also called the *top* of S and is denoted by ⊤.
- Dually, t∈S is said to be *minimal* in S iff there is no s∈S such that s≤t. A set may have any number of minimal elements, including zero.
- If t is S's only minimal element, then t is the *minimum* of S.
- The minimum element t (if it exists) is also called the *bottom* of S and is denoted by ⊥.

Examples:

1. p∧q∧r is a maximal element of the set in Figure 1.4.. Since it is the only maximal element, it is the maximum or top.
2. The set in Figure 1.4 has three minimal elements (p, q, and r). It has no minimum (because it has three minimal elements).
3. The set of all integers has no maximal or minimal elements (Figure 1). It has no maximum (because it has no maximal elements); similarly, it has no minimum.

1.3 PARTIALLY ORDERED SET

Especially order theory, a partially ordered set (also poset) formalizes and generalizes the intuitive concept of an ordering, sequencing, or arrangement of the elements of a set. A poset consists of a set together with a binary relation indicating that, for certain pairs of elements in the set, one of the elements precedes the other in the ordering. The relation itself is called a "partial order." The word *partial* in the names "partial order" and "partially ordered set" is used as an indication that not every pair of elements needs to be comparable. That is, there may be pairs of elements for which neither element precedes the other in the poset. Partial orders thus generalize total orders, in which every pair is comparable.

Formally, a partial order is any binary relation that is reflexive (each element is comparable to itself), antisymmetric (no two different elements precede each other), and transitive (the start of a chain of precedence relations must precede the end of the chain).

One familiar example of a partially ordered set is a collection of people ordered by genealogical descendancy. Some pairs of people bear the descendant-ancestor relationship, but other pairs of people are incomparable, with neither being a descendant of the other.

A poset can be visualized through its Hasse diagram, which depicts the ordering relation.

A (non-strict) partial order[2] is a binary relation $\leq$ over a set P satisfying particular axioms which are discussed below. When $a \leq b$, we say that a is related to b. (This does not imply that b is also related to a, because the relation need not be symmetric.)

The axioms for a non-strict partial order state that the relation $\leq$ is reflexive, antisymmetric, and transitive. That is, for all a, b, and c in P, it must satisfy:

1. $a \leq a$ (reflexivity: every element is related to itself).
2. if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry: two distinct elements cannot be related in both directions).
3. if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity: if a first element is related to a second element, and, in turn, that element is related to a third element, then the first element is related to the third element).

In other words, a partial order is an antisymmetric preorder.

A set with a partial order is called a partially ordered set (also called a poset). The term *ordered set* is sometimes also used, as long as it is clear from the context that no other kind of order is meant. In particular, totally ordered sets can also be referred to as "ordered sets", especially in areas where these structures are more common than posets.

For a, b, elements of a partially ordered set P, if $a \leq b$ or $b \leq a$, then a and b are comparable. Otherwise they are incomparable. In the figure on top-right, e.g. {x} and {x,y,z} are comparable, while {x} and {y} are not. A partial order under which every pair of elements is comparable is called a total order or linear order; a totally ordered set is also called a chain (e.g., the natural numbers with their standard order). A subset of a poset in which no two distinct elements are comparable is called an antichain (e.g. the set of singletons{{x}, {y}, {z}} in the top-right figure). An element *a* is said to be strictly less than an element b, if $a \leq b$ and $a \neq b$. An element *a* is said to be covered by another element *b*, written *a*<:*b*, if *a* is strictly less than *b* and no third element *c* fits between them; formally: if both $a \leq b$ and $a \neq b$ are true, and $a \leq c \leq b$ is false for each *c* with $a \neq c \neq b$. A more concise definition will be given below using the strict order corresponding to "≤". For example, {x} is covered by {x,z} in the top-right figure, but not by {x,y,z}.

Examples

Standard examples of posets arising in mathematics include:

- The real numbers ordered by the standard *less-than-or-equal* relation ≤ (a totally ordered set as well).
- The set of subsets of a given set (its power set) ordered by inclusion (see the figure on top-right). Similarly, the set of sequences ordered by subsequence, and the set of stringsordered by substring.
- The set of natural numbers equipped with the relation of divisibility.
- The vertex set of a directed acyclic graph ordered by reachability.
- The set of subspaces of a vector space ordered by inclusion.
- For a partially ordered set *P*, the sequence space containing all sequences of elements from *P*, where sequence *a* precedes sequence *b* if every item in *a* precedes the corresponding item in *b*. Formally, $(a_n)_{n\in\mathbb{N}} \leq (b_n)_{n\in\mathbb{N}}$ if and only if $a_n \leq b_n$ for all *n* in ℕ, i.e. a componentwise order.
- For a set *X* and a partially ordered set *P*, the function space containing all functions from *X* to *P*, where $f \leq g$ if and only if $f(x) \leq g(x)$ for all *x* in *X*.
- A fence, a partially ordered set defined by an alternating sequence of order relations $a < b > c < d$ ...
- The set of events in special relativity and, in most cases,[3] general relativity, where for two events X and Y, X ≤ Y if and only if Y is in

the future light cone of X. An event Y can only be causally affected by X if X ≤ Y.

1.3.1 Extrema

There are several notions of "greatest" and "least" element in a poset *P*, notably:

- Greatest element and least element: An element *g* in *P* is a greatest element if for every element *a* in *P*, $a \leq g$. An element *m* in *P* is a least element if for every element *a* in *P*, $a \geq m$. A poset can only have one greatest or least element.
- Maximal elements and minimal elements: An element *g* in P is a maximal element if there is no element *a* in *P* such that $a > g$. Similarly, an element *m* in *P* is a minimal element if there is no element *a* in P such that $a < m$. If a poset has a greatest element, it must be the unique maximal element, but otherwise there can be more than one maximal element, and similarly for least elements and minimal elements.
- Upper and lower bounds: For a subset *A* of *P*, an element *x* in *P* is an upper bound of *A* if $a \leq x$, for each element *a* in *A*. In particular, *x* need not be in *A* to be an upper bound of *A*. Similarly, an element *x* in *P* is a lower bound of *A* if $a \geq x$, for each element *a* in *A*. A greatest element of *P* is an upper bound of *P* itself, and a least element is a lower bound of *P*.

For example, consider the positive integers, ordered by divisibility: 1 is a least element, as it divides all other elements; on the other hand this poset does not have a greatest element (although if one would include 0 in the poset, which is a multiple of any integer, that would be a greatest element; see figure). This partially ordered set does not even have any maximal elements, since any *g* divides for instance 2*g*, which is distinct from it, so *g* is not maximal. If the number 1 is excluded, while keeping divisibility as ordering on the elements greater than 1, then the resulting poset does not have a least element, but any prime number is a minimal element for it. In this poset, 60 is an upper bound (though not a least upper bound) of the subset {2,3,5,10}, which does not have any lower bound (since 1 is not in the poset); on the other hand 2 is a lower bound of the subset of powers of 2, which does not have any upper bound.

1.3.2 Orders on the Cartesian product of partially ordered sets

In order of increasing strength, i.e., decreasing sets of pairs, three of the possible partial orders on the Cartesian product of two partially ordered sets are (see figures):

- the lexicographical order: $(a,b) \leq (c,d)$ if $a < c$ or ($a = c$ and $b \leq d$);
- the product order: $(a,b) \leq (c,d)$ if $a \leq c$ and $b \leq d$;
- the reflexive closure of the direct product of the corresponding strict orders: $(a,b) \leq (c,d)$ if ($a < c$ and $b < d$) or ($a = c$ and $b = d$).

All three can similarly be defined for the Cartesian product of more than two sets.

Applied to ordered vector spaces over the same field, the result is in each case also an ordered vector space.

1.3.3 Sums of partially ordered sets

Another way to combine two posets is the **ordinal sum**[4] (or **linear sum**[5]), $Z = X \oplus Y$, defined on the union of the underlying sets X and Y by the order $a \leq_Z b$ if and only if:

- $a, b \in X$ with $a \leq_X b$, or
- $a, b \in Y$ with $a \leq_Y b$, or
- $a \in X$ and $b \in Y$.

If two posets are well-ordered, then so is their ordinal sum.[6] The ordinal sum operation is one of two operations used to form series-parallel partial orders, and in this context is called series composition. The other operation used to form these orders, the disjoint union of two partially ordered sets (with no order relation between elements of one set and elements of the other set) is called in this context parallel composition.

1.3.4 Strict and non-strict partial orders

In some contexts, the partial order defined above is called a non-strict (or reflexive) partial order. In these contexts, a strict(or irreflexive) partial order "<" is a binary relation that is irreflexive, transitive and asymmetric, i.e. which satisfies for all *a*, *b*, and *c* in *P*:

- not $a < a$ (irreflexivity),
- if $a < b$ and $b < c$ then $a < c$ (transitivity), and
- if $a < b$ then not $b < a$ (asymmetry; implied by irreflexivity and transitivity[7]).

Strict and non-strict partial orders are closely related. A non-strict partial order may be converted to a strict partial order by removing all relationships of the form $a \leq a$. Conversely, a strict partial order may be converted to a non-strict partial order by adjoining all relationships of that form. Thus, if "≤" is a non-strict partial order, then the corresponding strict partial order "<" is the irreflexive kernel given by:

$a < b$ if $a \leq b$ and $a \neq b$

Conversely, if "<" is a strict partial order, then the corresponding non-strict partial order "≤" is the reflexive closure given by:

$a \leq b$ if $a < b$ or $a = b$.

This is the reason for using the notation "≤".

Using the strict order "<", the relation "*a* is covered by *b*" can be equivalently rephrased as "$a<b$, but not $a<c<b$ for any *c*". Strict partial orders are useful because they correspond more directly to directed acyclic graphs (dags): every strict partial order is a

dag, and the transitive closure of a dag is both a strict partial order and also a dag itself.

1.3.5 Mappings between partially ordered sets

Given two partially ordered sets $(S,\leq)$ and $(T,\leq)$, a function $f: S \to T$ is called order-preserving, or monotone, or isotone, if for all x and y in S, $x\leq y$ implies $f(x) \leq f(y)$. If $(U,\leq)$ is also a partially ordered set, and both $f: S \to T$ and $g: T \to U$ are order-preserving, their composition$(g\circ f): S \to U$ is order-preserving, too. A function $f: S \to T$ is called order-reflecting if for all x and y in S, $f(x) \leq f(y)$ implies $x\leq y$. If f is both order-preserving and order-reflecting, then it is called an order-embedding of $(S,\leq)$ into $(T,\leq)$. In the latter case, f is necessarily injective, since $f(x) = f(y)$ implies $x \leq y$ and $y \leq x$. If an order-embedding between two posets S and T exists, one says that S can be embedded into T. If an order-embedding $f: S \to T$ is bijective, it is called an order isomorphism, and the partial orders $(S,\leq)$ and $(T,\leq)$ are said to be isomorphic. Isomorphic orders have structurally similar Hasse diagrams (cf. right picture). It can be shown that if order-preserving maps $f: S \to T$ and $g: T \to S$ exist such that $g\circ f$ and $f\circ g$ yields the identity functionon S and T, respectively, then S and T are order-isomorphic.

For example, a mapping $f: \mathbb{N} \to \mathbb{P}(\mathbb{N})$ from the set of natural numbers (ordered by divisibility) to the power set of natural numbers (ordered by set inclusion) can be defined by taking each number to the set of its prime divisors. It is order-preserving: if x divides y, then each prime divisor of x is also a prime divisor of y. However, it is neither injective (since it maps both 12 and 6 to {2,3}) nor order-reflecting (since besides 12 doesn't divide 6). Taking instead each number to the set of its prime power divisors defines a map $g: \mathbb{N} \to \mathbb{P}(\mathbb{N})$ that is order-preserving, order-reflecting, and hence an order-embedding. It is not an order-isomorphism (since it e.g. doesn't map any number to the set {4}), but it can be made one by restricting its codomain to $g(\mathbb{N})$. The right picture shows a subset of $\mathbb{N}$ and its isomorphic image under g. The construction of such an order-isomorphism into a power set can be generalized to a wide class of partial orders, called distributive lattices, see "Birkhoff's representation theorem".

1.3.6 Well Ordered Set

A well-order (or well-ordering or well-order relation) on a set *S* is a total order on *S* with the property that every non-empty subset of *S* has a least element in this ordering. The set *S* together with the well-order relation is then called a well-ordered set. In some academic articles and textbooks these terms are instead written as wellorder, wellordered, and wellordering or well order, well ordered, and well ordering.

Every non-empty well-ordered set has a least element. Every element *s* of a well-ordered set, except a possible greatest element, has a unique successor (next element), namely the least element of the subset of all elements greater than *s*. There may be elements besides the least element which have no predecessor (see *Natural numbers* below for an example). In a well-ordered set *S*, every subset *T* which has an upper bound has a least upper bound, namely the least element of the subset of all upper bounds of *T* in *S*.

If $\leq$ is a non-strict well ordering, then $<$ is a strict well ordering. A relation is a strict well ordering if and only if it is a well-founded strict total order. The distinction between strict and non-strict well orders is often ignored since they are easily interconvertible.

Every well-ordered set is uniquely order isomorphic to a unique ordinal number, called the order type of the well-ordered set. The well-ordering theorem, which is equivalent to the axiom of choice, states that every set can be well ordered. If a set is well ordered (or even if it merely admits a well-founded relation), the proof technique of transfinite induction can be used to prove that a given statement is true for all elements of the set.

The observation that the natural numbers are well ordered by the usual less-than relation is commonly called the well-ordering principle (for natural numbers).

Check in Progress-II

Note: i) Write your answers in the space given below.

Q. 1 Define Well Ordered Set.

Solution

…………………………………………………………………………..

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

Q. 2 Define Extrema.

Solution

…………………………………………………………………………..

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

1.4 ORDINAL NUMBERS

Every well-ordered set is uniquely order isomorphic to a unique ordinal number, called the order type of the well-ordered set. The position of each element within the ordered set is also given by an ordinal number. In the case of a finite set, the basic operation of counting, to find the ordinal number of a particular object, or to find the object with a particular ordinal number, corresponds to assigning ordinal numbers one by one to the objects. The size (number of elements, cardinal number) of a finite set is equal to the order type. Counting in the everyday sense typically starts from one, so it assigns to each object the size of the initial segment with that object as last element. Note that these numbers are one more than the formal ordinal numbers according to the isomorphic order, because these are equal to the number of earlier objects (which corresponds to counting from zero). Thus for finite *n*, the expression "*n*-th element" of a well-ordered set requires context to know whether this counts from zero or one. In a notation "β-th element" where β can also be an infinite ordinal, it will typically count from zero.

For an infinite set the order type determines the cardinality, but not conversely: well-ordered sets of a particular cardinality can have many different order types. For a countably infinite set, the set of possible order types is even uncountable.

1.4.1 Natural numbers

The standard ordering ≤ of the natural numbers is a well ordering and has the additional property that every non-zero natural number has a unique predecessor.

Another well ordering of the natural numbers is given by defining that all even numbers are less than all odd numbers, and the usual ordering applies within the evens and the odds:

0 2 4 6 8 ... 1 3 5 7 9 ...

This is a well-ordered set of order type $\omega + \omega$. Every element has a successor (there is no largest element). Two elements lack a predecessor: 0 and 1.

1.4.2 Integers

Unlike the standard ordering ≤ of the natural numbers, the standard ordering ≤ of the integers is not a well ordering, since, for example, the set of negative integers does not contain a least element.

The following relation R is an example of well ordering of the integers: $x\,R\,y$ if and only if one of the following conditions holds:

1. $x = 0$
2. x is positive, and y is negative
3. x and y are both positive, and $x \leq y$
4. x and y are both negative, and $|x| \leq |y|$

This relation R can be visualized as follows:

0 1 2 3 4 ... −1 −2 −3 ...

R is isomorphic to the ordinal number $\omega + \omega$.

Another relation for well ordering the integers is the following definition: $x \leq_z y$ iff ($|x| < |y|$ or ($|x| = |y|$ and $x \leq y$)). This well order can be visualized as follows:

Notes

$0\ {-1}\ 1\ {-2}\ 2\ {-3}\ 3\ {-4}\ 4\ ...$

This has the order type ω.

1.4.3 Reals

The standard ordering ≤ of any real interval is not a well ordering, since, for example, the open interval (0, 1) ⊆ [0,1] does not contain a least element. From the ZFC axioms of set theory (including the axiom of choice) one can show that there is a well order of the reals. Also Wacław Sierpiński proved that ZF + GCH (the generalized continuum hypothesis) imply the axiom of choice and hence a well order of the reals. Nonetheless, it is possible to show that the ZFC+GCH axioms alone are not sufficient to prove the existence of a definable (by a formula) well order of the reals.[1] However it is consistent with ZFC that a definable well ordering of the reals exists—for example, it is consistent with ZFC that V=L, and it follows from ZFC+V=L that a particular formula well orders the reals, or indeed any set.

An uncountable subset of the real numbers with the standard ordering ≤ cannot be a well order: Suppose X is a subset of R well ordered by ≤. For each x in X, let $s(x)$ be the successor of x in ≤ ordering on X (unless x is the last element of X). Let $A = \{ (x, s(x)) \mid x \in X \}$ whose elements are nonempty and disjoint intervals. Each such interval contains at least one rational number, so there is an injective function from A to Q. There is an injection from X to A (except possibly for a last element of X which could be mapped to zero later). And it is well known that there is an injection from Q to the natural numbers (which could be chosen to avoid hitting zero). Thus there is an injection from X to the natural numbers which means that X is countable. On the other hand, a countably infinite subset of the reals may or may not be a well order with the standard "≤". For example,

- The natural numbers are a well order under the standard ordering ≤.
- The set {1/n : n =1,2,3,...} has no least element and is therefore not a well order under standard ordering ≤.

Examples of well orders:

- The set of numbers $\{ -2^{-n} \mid 0 \leq n < \omega \}$ has order type ω.
- The set of numbers $\{ -2^{-n} - 2^{-m-n} \mid 0 \leq m,n < \omega \}$ has order type ω². The previous set is the set of limit points within the set. Within the set of real numbers, either with the ordinary topology or the order topology, 0 is also a limit point of the set. It is also a limit point of the set of limit points.
- The set of numbers $\{ -2^{-n} \mid 0 \leq n < \omega \} \cup \{ 1 \}$ has order type ω + 1. With the order topology of this set, 1 is a limit point of the set. With the ordinary topology (or equivalently, the order topology) of the real numbers it is not.

1.5 SUMMARY

We study in this units about ordered set and its proposition and properties. We study Strict and Non-Strict partial Ordered set. We study Mapping b/w partial ordered set. Se study axiom of choice function.

1.6 KEYWORD

Partial : Existing only in part; incomplete

Axiom : A statement or proposition which is regarded as being established, accepted, or self-evidently true.

1.7 EXERCISE

Q. 1 What is well order Sets give example of well order sets?

Q. 2 Find existence of choice function with axiom .

Q. 3 What is strict and non-strict partial ordered sets. ?

Q. 4 Give example of ordinals numbers.

Q. 5 Understanding mpping between partial ordered sets .

1.8 ANSWER FOR CHECK IN PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 1.5

Q 2 Check in Section 1.3

Check in Progress-II

Answer Q. 1 Check in Section 3.6

Q 2 Check in Section 3.1

1.9 SUGGESTION READING AND REFERENCES

- *Halmos, Paul R. (1960). Naive Set Theory. The University Series in Undergraduate Mathematics. Princeton, NJ: van Nostrand Company. Zbl 0087.04403.*
- *Herrlich, Horst (2006). Axiom of Choice. Lecture Notes in Math. 1876. Berlin: Springer-Verlag. ISBN 978-3-540-30989-5.*
- *Howard, Paul; Rubin, Jean E. (1998). Consequences of the axiom of choice. Mathematical Surveys and Monographs. 59. Providence, Rhode Island: American Mathematical Society. ISBN 9780821809778.*
- *Jech, Thomas (2008) [1973]. The axiom of choice. Mineola, New York: Dover Publications. ISBN 978-0-486-46624-8.*
- Thomas Jech, "About the Axiom of Choice." *Handbook of Mathematical Logic*, John Barwise, ed., 1977.
- *Lévy, Azriel (1958). "The independence of various definitions of finiteness" (PDF). Fundamenta Mathematicae. 46: 1–13.*
- Per Martin-Löf, "100 years of Zermelo's axiom of choice: What was the problem with it?", in *Logicism, Intuitionism, and Formalism: What Has Become of Them?*, Sten Lindström, Erik Palmgren, Krister Segerberg, and Viggo Stoltenberg-Hansen, editors (2008). ISBN 1-4020-8925-2
- *Mendelson, Elliott (1964). Introduction to Mathematical Logic. New York: Van Nostrand Reinhold.*
- *Moore, Gregory H. (1982). Zermelo's axiom of choice, Its origins, development and influence. Springer. ISBN 978-0-387-90670-6.,*

available as a Dover Publications reprint, 2013, ISBN 0-486-48841-1.

- *Feferman, S. (1964). "Some Applications of the Notions of Forcing and Generic Sets". Fundamenta Mathematicae. 56 (3): 325–345.*

- *Folland, Gerald B. (1999). Real Analysis: Modern Techniques and Their Applications. Pure and applied mathematics (2nd ed.). Wiley. pp. 4–6, , 9. ISBN 978-0-471-31716-6.*

`

UNIT 2 ORDERED SET –II

STRUCTURE

2.0 OBJECTIVE

- Here after study this unit we are able to know ordered structure
- Learn minimal uncountable well-ordered set
- Learn algebra of ordinalities
- Learn Zorn's Lemma
- Learn Ordering Ordinalities

2.1 INTRODUCTION

A **well-ordered set** is a set with an order such that every its nonempty subset has a smallest element.

CONSTRUCTING WELL-ORDERED SETS, WELL-ORDERING THEOREM

There are several ways of constructing well-ordered sets, for example:

1. A subset of a well-ordered set is well-ordered.
2. The product of a finite number of well-ordered sets can be well-ordered (in the dictionary order, for example).
3. The union of an arbitrary collection of disjoint well-ordered sets indexed by a well-ordered set can be well-ordered (compare first indexes, then elements).
4. Every nonempty finite ordered set has the order type of a section of Z+ , so is well-ordered.

Further, assuming the axiom of choice, every set can be well-ordered.

2.2 WELL-ORDERING THEOREM

(**Well-Ordering Theorem**) For every set there is an order on it that is a well-ordering.

This theorem was proved by Zermelo in 1904, and it startled the mathematical world. There was considerable debate as to the correctness of the proof... When the proof was analyzed closely, the only point at which it was found that there might be some question was a construction involving an infinite number of arbitrary choices, that is, a construction involving — the choice axiom. Some mathematicians rejected the choice axiom as a result, and for many years a legitimate question about a new theorem was: Does its proof involve the choice axiom or not?.. Present-day mathematicians, by and large, do not have such qualms. They accept the axiom of choice as a reasonable assumption about set theory, and they accept the well-ordering theorem along with it.

In fact, neither accepting nor rejecting the axiom of choice leads to a contradiction. This is purely a matter of choice — which math universe is more suitable for the current purposes.

- The axiom of choice is equivalent to the well-ordering theorem, see Supplementary Exercises.
- A weaker result is of primary interest: there exists an uncountable well-ordered set.

EXAMPLES

Minimal uncountable well-ordered set

A section S_α of a well-ordered set is defined by $S_\alpha=\{x|x<\alpha\}$.

2.2.1 Minimal uncountable Well-Ordered set:

An uncountable well-ordered set S_Ω every section of which is countable. There exists such a set, and its order type is uniquely determined by the definition. $S_\Omega \cup \{\Omega\}$ is denoted by $\overline{S}_\Omega$.

The proof of the existence of S_Ω uses the assumption that there exists an uncountable well-ordered set.

- Every countable subset of S_Ω has an upper bound.
- S_Ω has no largest element.
- Every element of S_Ω has an immediate successor.
- There are uncountably many elements in S_Ω having no immediate predecessor.

Antidictionary order

Let $A \subset (Z_+)^\omega$ be the set of all sequences that are eventually 1 . Then, the **antidictionary order**, which prescribes to compare sequences "from right to left", i.e. by the last element in which the sequences differ, is a well-ordering on A .

FACTS

Let A be an ordered set.

- If A is well-ordered, then it has the least upper bound property.
- If A is well-ordered, then every $a \in A$ except for the largest (if exists) has an immediate successor.
- A is not well-ordered iff it has a countable subset having the same order type as Z_- .
    - A is well-ordered iff every countable subset of A is well-ordered.

2.2.2 Principle of Transfinite Induction & General Principle of Recursive

The Principle of Transfinite Induction. Let A be a well-ordered set. $B \subset A$ is called **inductive** if for every $\alpha \in A$, $S\alpha \subset B$ implies $\alpha \in B$. If B is inductive, then $B=A$ (compare to the strong induction principle, Section 4).

The General Principle of Recursive Definition. (*see Supplementary Exercises*) Let A be a well-ordered set and B be a set. Further, let F be the set of all functions from all sections of A to B , and $\rho:F\to B$ be a recursive rule. Then, there exists a unique function h such that for $\alpha \in A$: $h(\alpha)=\rho(h|S\alpha)$ (compare to the principle of recursive definition, Section 8).

CARDINALITY

For every two sets A and B either they have the same cardinality, or one has the cardinality greater than the other (assuming the well-ordering theorem).

- Indeed, either there is a surjection from A onto B , implying there is an injection from B into A , or there is no surjection, and we can well-order the sets, and by the general principle of recursive definition define an injective function h from A to B using $\rho(f:S\alpha\to B)=\text{smallest}[B-f(S\alpha)]$, in which case $h(\alpha)=\text{smallest}[B-h(S\alpha)]$.

2.3 WELL-ORDERED SETS AND ORDINALITIES.

Exercise 3.1: Show that for a linearly ordered set X, TFAE: (i) X satisfies the descending chain condition: there are no infinite strictly descending sequences x1 > x2 > . . . in X. (ii) X is well-ordered. We need the notion of “equivalence” of well-ordered sets. A mapping f : S → T between partially ordered sets is order preserving (or an order homomorphism) if $s1 \leq s2$ in S implies $f(s1) \leq f(s2)$ in T.

Exercise 3.2 : Let $f : S \to T$ and $g : T \to U$ be order homomorphisms of partially ordered sets. a) Show that $g \circ f : S \to U$ is an order homomorphism.

b) Note that the identity map from a partially ordered set to itself is an order homomorphism. (It follows that there is a category whose objects are partially ordered sets and whose morphisms are order homomorphisms.) An order isomorphism between posets is a mapping f which is order preserving, bijective, and whose inverse f^{-1} is order preserving. (This is the general – i.e., categorical – definition of isomorphism of structures.)

Exercise 3.3: Give an example of an order preserving bijection f such that f^{-1} is not order preserving. However:

Lemma 1. An order-preserving bijection whose domain is a totally ordered set is an order isomorphism.

Exercise 3.4: Prove Lemma 1. Lemma 2. (Rigidity Lemma) Let S and T be well-ordered sets and $f1, f2 : S \to T$ two order isomorphisms. Then $f1 = f2$. Proof: Let f1 and f2 be two order isomorphisms between the well-ordered sets S and T, which we may certainly assume are nonempty. Consider S2, the set of elements s of S such that $f1(s) \neq f2(s)$, and let $S1 = S \setminus S2$. Since the least element of S must get mapped to the least element of T by any order-preserving map, S1 is nonempty; put $T1 = f1(S1) = f2(S1)$. Supposing that S2 is nonempty, let s2 be its least element. Then f1(s2) and f2(s2) are both characterized by being the least element of $T \setminus T1$, so they must be equal, a contradiction.

Exercise 3.5: Let S be a partially ordered set.

a) Show that the order isomorphisms $f : S \to S$ form a group, the order automorphism group Aut(S) of S. (The same holds for any object in any category.)

b) Notice that Lemma 2 implies that the automorphism group of a well-ordered set is the trivial group.1

c) Suppose S is linearly ordered and f is an order automorphism of S such that for some positive integer n we have $f^n = Id_S$, the identity map. Show that $f = Id_S$. (Thus the automorphism group of a linearly ordered set is torsionfree.)

d) For any infinite cardinality κ, find a linearly ordered set S with $|\text{Aut}(S)| \geq \kappa$. Can we always ensure equality?

e)** Show that every group G is (isomorphic to) the automorphism group of some partially ordered set.

Let us define an ordinality to be an order-isomorphism class of well-ordered sets, and write o(X) for the order-isomorphism class of X. The intentionally graceless terminology will be cleaned up later on. Since two-order isomorphic sets are equipotent, we can associate to every ordinality α an "underlying" cardinality $|\alpha|$: $|o(X)| = |X|$. It is natural to expect that the classification of ordinalities will be somewhat richer than the classification of cardinalities – in general, endowing a set with extra structure leads to a richer classification – but the reader new to the subject may be (we hope, pleasantly) surprised at how much richer the theory becomes.

From the perspective of forming "isomorphism classes" (a notion the ontological details of which we have not found it profitable to investigate too closely) ordinalities have a distinct advantage over cardinalities: according to the Rigidity Lemma, any two representatives of the same ordinality are uniquely (hence canonically!) isomorphic. This in turn raises the hope that we can write down a canonical representative of each ordinality. This hope has indeed been realized, by von Neumann, as we shall see later on: the canonical representatives will be called "ordinals." While we are alluding to later developments, let us mention that just as we can associate a cardinality to each ordinality, we can also – and this is much more profound – associate an ordinality o(κ) to each cardinality κ. This assignment is one-to-one, and this allows us to give a canonical representative to each cardinality, a "cardinal." At least at the current level of discussion, there is no purely mathematical advantage to the passage from cardinalities to cardinals, but it has a 1One says that a structure is rigid if it has no nontrivial automorphisms. striking ontological2 consequence, namely that, up to isomorphism, we may develop all of set theory in the context of "pure sets", i.e., sets whose elements (and whose elements' elements, and . . .) are themselves sets. But first let us give some basic examples of ordinalities and ways to construct new ordinalities from preexisting ones.

2.4 ALGEBRA OF ORDINALITIES.

Example 4.1: Trivially the empty set is well-ordered, as is any set of cardinality one. These sets, and only these sets, have unique well-orderings.

Example 4.2: Our "standard" example [n] of the cardinality n comes with a well-ordering. Moreover, on a finite set, the concepts of well-ordering and linear ordering coincide, and it is clear that there is up to order isomorphism a unique linear ordering on [n]. Informally, given any two orderings on an n element set, we define an order-preserving bijection by pairing up the least elements, then the second-least elements, and so forth. (For a formal proof, use induction.)

Example 4.3: The usual ordering on N is a well-ordering. Notice that this is isomorphic to the ordering on $\{n \in Z \mid n \geq n0\}$ for any $n0 \in Z$. As is traditional, we write ω for the ordinality of N. Exercise 1.2.4: For any ordering ≤ on a set X, we have the opposite ordering ≤′ , defined by $x \leq' y$ iff $y \leq x$. a) If ≤ is a linear ordering, so is ≤′ . b) If both ≤ and ≤′ are well-orderings, then X is finite. For a partially ordered set X, we can define a new partially ordered set $X+ := X \cup \{\infty\}$ by adjoining some new element ∞ and decreeing $x \leq \infty$ for all $x \in X$.

Proposition 3. If X is well-ordered, so is X+.

Proof: Let Y be a nonempty subset of X+. Certainly there is a least element if |Y | = 1; otherwise, Y contains an element other than ∞, so that Y ∩X is nonempty, and its least element will be the least element of Y .

If X and Y are order-isomorphic, so too are X+ and Y +, so the passage from X to X+ may be viewed as an operation on ordinalities. We denote o(X+) by o(X) + 1, the successor ordinality of o(X).

Note that all the finite ordinalities are formed from the empty ordinality 0 by iterated successorship. However, not every ordinality is of the form o ′ + 1, e.g. ω is clearly not: it lacks a maximal element. (On the other hand, it is obtained from all the finite ordinalities in a way that we will come back to shortly.) We will say that an ordinality o is a successor ordinality if it is of the form o ′ + 1 for some ordinality o ′ and a limit ordinality otherwise. Thus 0 and ω are limit ordinals. 2 I restrain myself from writing "ontological" (i.e., with quotation marks), being like most contemporary mathematicians alarmed by statements about the reality of mathematical objects.

Example 4.4: The successor operation allows us to construct from ω the new ordinals ω + 1, ω + 2 := (ω + 1) + 1, and for all n ∈ Z +, ω + n := (ω + (n − 1)) + 1: these are all distinct ordinals. Definition: For partially ordered sets (S1, ≤1) and (S2, ≤2), we define S1 + S2 to be the set S1 ⊔ S2 with s ≤ t if either of the following holds: (i) For i = 1 or 2, s and t are both in Si and s ≤i t; (ii) s ∈ S1 and s ∈ S2. Informally, we may think of S1 + S2 as "S1 followed by S2."

Proposition 4. If S1 and S2 are linearly ordered (resp. well-ordered), so is S1+S2

. **Exercise 4.5: Prove Proposition 4.**

Again the operation is well-defined on ordinalities, so we may speak of the ordinal sum o+o ′ . By taking S2 = [1], we recover the successor ordinality: o+ [1] = o+ 1.

Example 4.6: The ordinality 2ω := ω + ω is the class of a well-ordered set which contains one copy of the natural numbers followed by another.

Proceeding inductively, we have $n\omega := (n - 1)\omega + \omega$, with a similar description.

Tournant Dangereuse: We can also form the ordinal sum $1 + \omega$, which amounts to adjoining to the natural numbers a smallest element. But this is still orderisomorphic to the natural numbers: $1 + \omega = \omega$. In fact the identity $1 + o = o$ holds for every infinite ordinality (as will be clear later on). In particular $1 + \omega \neq \omega + 1$, so beware: the ordinal sum is not commutative! (To my knowledge it is the only non-commutative operation in all of mathematics which is invariably denoted by "+".) It is however immediately seen to be associative.

The notation 2ω suggests that we should have an ordinal product, and indeed we do:

Definition: For posets ($S1$, ≤ 1) and ($S2$, ≤ 2) we define the lexicographic product to be the Cartesian product $S1 \times S2$ endowed with the relation $(s1, s2) \leq (t1, t2)$ if(f) either $s1 \leq t1$ or $s1 = t1$ and $s2 \leq t2$. If the reasoning behind the nomenclature is unclear, I suggest you look up "lexicographic" in the dictionary.3 Proposition 5. If S1 and S2 are linearly ordered (resp. well-ordered), so is $S1 \times S2$.

Check in Progress-I

Q. 1 State Well Ordered Set.

Solution

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

Q. 2 Define Ordanilities .

Solution

…………………………………………………………………………

……………………………………………………………………………

………………………………………………………………………………

……………………………………………………………………………

Exercise 4.7: Prove Proposition 5.

As usual this is well-defined on ordinalities so leads to the ordinal product $o \cdot o'$.

Example 4.8: For any well-ordered set X, [2]·X gives us one copy {(1, x) | x ∈ X} followed by another copy {(2, x) | x ∈ X}, so we have a natural isomorphism of $[2] \cdot X$ with $X + X$ and hence $2 \cdot o = o + o$. (Similarly for 3o and so forth.) This time we should be prepared for the failure of commutativity: $\omega \cdot n$ is isomorphic to ω. This allows us to write down $\omega 2 := \omega \times \omega$, which we visualize by starting with the positive integers and then "blowing up" each positive integer to give a whole order isomorphic copy of the positive integers again. Repeating this operation gives $\omega 3 = \omega 2 \cdot \omega$, and so forth. Altogether this allows us to write down ordinalities of the form $P(\omega) = an\omega\ n +. . .+a1\omega +a0$ with $ai \in N$, i.e., polynomials in ω with natural number coefficients. It is in fact the case that (i) distinct polynomials $P \neq Q \in N[T]$ give rise to distinct ordinalities $P(\omega) \neq Q(\omega)$; and (ii) any ordinality formed from [n] and ω by finitely many sums and products is equal to some P(ω) – even when we add/multiply in "the wrong order", e.g. $\omega * 7 * \omega 2 * 4 + \omega * 3 + 11 = \omega 3 + \omega + 11$ – but we will wait until we know more about the ordering of ordinalities to try to establish these facts.

Example 4.9: Let $\alpha 1 = o(X1), . . . , \alpha n = o(Xn)$ be ordinalities.

a) Show that $\alpha 1 \times (\alpha 2 \times \alpha 3)$ and $(\alpha 1 \times \alpha 2) \times \alpha 3$ are each order isomorphic to the set $X1 \times X2 \times X3$ endowed with the ordering $(x1, x2, x3) \leq (y1, y2, y3)$ if $x1 < y1$ or ($x1 = y1$ and ($x2 < y2$ or ($x2 = y2$ and $x3 \leq y3$))). In particular ordinal multiplication is associative.

b) Give an explicit definition of the product well-ordering on $X1\times. . .\times Xn$, another "lexicographic ordering."

In fact, we also have a way to exponentiate ordinalities: let $\alpha = o(X)$ and $\beta = o(Y)$. Then by α β we mean the order isomorphism class of the set Z = Z(Y, X) of all functions $f : Y \to X$ with $f(y) = 0X$ (0X denotes the minimal element of X) for all but finitely many $y \in Y$, ordered by $f1 \leq f2$ if $f1 = f2$ or, for the greatest element $y \in Y$ such that $f1(y) \neq f2(y)$ we have $f1(y) < f2(y)$.

Some helpful terminology: one has the zero function, which is 0 for all values. For every other $f \in W$, we define its degree ydeg to be the largest $y \in Y$ such that $f(y) \neq 0$ and its leading coefficient xl := f(ydeg).

Proposition 6. For ordinalities α and β, α β is an ordinality.

Proof: Let Z be the set of finitely nonzero functions $f : Y \to X$ as above, and let $W \subset Z$ be a nonempty subset. We may assume 0 is not in W, since the zero function is the minimal element of all of Z. Thus the set of degrees of all elements of W is nonempty, and we may choose an element of minimal degree y1; moreover, among all elements of minimal degree we may choose one with minimal leading coefficient x1, say f1. Suppose f1 is not the minimal element of W, i.e., there exists $f' \in W2$ with $f' < f1$. Any such f' has the same degree and leading coefficient as f1, so the last value y' at which f' and f1 differ must be less than y1. Since f1 is nonzero at all such y' and f1 is finitely nonzero, the set of all such y' is finite and thus has a maximal element y2. Among all f' with $f'(y2) < f(y2)$ and $f'(y) = f(y)$ for all $y > y2$, choose one with $x2 = f'(y2)$ minimal and call it f2. If f2 is not minimal, we may continue in this way, and indeed get a sequence of elements $f1 > f2 > f3 \ldots$ as well as a descending chain $y1 > y2 > \ldots$. Since Y is well-ordered, this descending chain must terminate at some point, meaning that at some point we find a minimal element fn of W.

Example 4.10: The ordinality ω ω is the set of all finitely nonzero functions $f : N \to N$. At least formally, we can identify such functions as polynomials in ω with N-coefficients: $Pf(\omega) = \sum n \in N\ f(n)\omega\ n$. The well-ordering makes $Pf < Pg$ if the at the largest n for which $f(n) \neq g(n)$ we have $f(n) < g(n)$, e.g. 6 PETE L. CLARK $\omega\ 3 + 2\omega\ 2 + 1 > \omega 3 + \omega\ 2 + \omega + 100$. It is hard to ignore the following observation: ω ω puts a natural well-ordering relation on all the ordinalities we had already defined. This makes us look back and see that the same seems to be the case for all ordinalities: e.g. ω itself is order isomorphic to the set of all the finite ordinalities [n] with the obvious order relation between them. Now that we see the suggested order relation on the ordinalities of the form P(ω) one can check that this is the case for them as well: e.g. ω 2 can be realized as the set of all linear polynomials {aω + b | a, b ∈ N}. This suggests the following line of inquiry:

(i) Define a natural ordering on ordinalities (as we did for cardinalities).

(ii) (ii) Show that this ordering well-orders any set of ordinalities.

Exercise 4.11: Let α and β be ordinalities.

a) Show that 0β = 0, 1β = 1, α 0 = 1, α 1 = α.

b) Show that the correspondence between finite ordinals and natural numbers respects exponentiation.

c) For an ordinal α, the symbol α n now has two possible meanings: exponentiation and iterated multiplication. Show that the two ordinalities are equal. (The proof requires you to surmount a small left-to-right lexicographic difficulty.) In particular |α n| = |α| n = |α|.

d) For any infinite ordinal β, show that |α β | = max(|α|, |β|)

Tournant dangereuse: In particular, it is generally not the case that |α β | = |α| |β| : e.g. 2ω and ω ω are both countable ordinalities. In fact, we have not yet seen any uncountable well-ordered sets, and one cannot construct an uncountable ordinal from ω by any finite iteration of the ordinal operations we have described (nor by a countable iteration either, although we have not yet made formal sense of that). This leads us to wonder: are there any uncountable ordinalities?

5 Ordering ordinalities. Let S1 and S2 be two well-ordered sets. In analogy with our operation ≤ on sets, it would seem natural to define S1 ≤ S2 if there exists an order-preserving injection S1 → S2. This gives a relation ≤ on ordinalities which is clearly symmetric and transitive.

However, this is not the most useful definition of ≤ for well-ordered sets, since it gives up the rigidity property. In particular, recall Dedekind's characterization of infinite sets as those which are in bijection with a proper subset of themselves, or, equivalently, those which inject into a proper subset of themselves. With the above definition, this will still occur for infinite ordinalities: for instance, we can inject ω properly into itself just by taking N → N, n 7→ n + 1. Even if we require the least elements to be preserved, then we can still inject N into any infinite subset of itself containing 0.

So as a sort of mild deus ex machina we will work with a more sophisticated order relation. First, for a linearly ordered set S and s ∈ S, we define

$I(s) = \{t \in S \mid t < s\}$,

an initial segment of S. Note that every initial segment is a proper subset. Let us also define

$I[s] = \{t \in S \mid t \leq s\}$.

Now, given linearly ordered sets S and T, we define S < T if there exists an order-preserving embedding f : S → T such that f(S) is an initial segment of T (say, an initial embedding). We define S ≤ T if S < T or S ∼= T.

Exercise 5.1: Let f : S1 → S2 and g : T1 → T2 be order isomorphisms of linearly ordered sets. a) Suppose s ∈ S1. Show that f(I(s)) = I(f(s)) and f(I[s])) = I(f[s]). b) Suppose that S1 < T1 (resp. S1 ≤ T1). Show that S2 < T2) (resp. S2 ≤ T2). c) Deduce that < and ≤ give well-defined relations on any set F of ordinalities.

Exercise 5.2: a) Show that if i : X → Y and j : Y → Z are initial embeddings of linearly ordered sets, then j ◦ i : X → Z is an initial embedding. b) Deduce that the relation < on any set of ordinalities is transitive.

Definition: In a partially ordered set X, a subset Z is an order ideal if for all z ∈ Z and x ∈ X, if x < z then x ∈ Z. For example, the empty set and X itself are always order ideals. We say that X is an improper order ideal of itself, and all other order ideals are proper. For instance, I[s] is an order ideal, which may or may not be an initial segment.

Lemma 7. ("Principal ideal lemma") Any proper order ideal in a well-ordered set is an initial segment.

Proof: Let Z be a proper order ideal in X, and s the least element of X \ Z. Then a moment's thought gives Z = I(s).

The following is a key result:

Theorem 8. (Ordinal trichotomy) For any two ordinalities $\alpha = o(X)$ and $\beta = o(Y)$, exactly one of the following holds: $\alpha < \beta$, $\alpha = \beta$, $\beta < \alpha$.

Corollary 9. Any set of ordinalities is linearly ordered under ≤.

Exercise 5.3: Deduce Corollary 9 from Theorem 8. Is the Corollary equivalent to the Theorem?

Proof of Theorem 8: Part of the assertion is that no well-ordered set X is order isomorphic to any initial segment I(s) in X (we would then have both $o(I(s)) < o(X)$ and $o(I(s)) = o(X)$); let us establish this first. Suppose to the contrary that $\iota : X \to X$ is an order embedding whose image is an initial segment I(s). Then the set of x for which $\iota(x) \neq x$ is nonempty (otherwise ι would be the identity map, and no linearly ordered set is equal to any of its initial segments), so let x be the least such element. Then, since ι restricted to I(x) is the identity map, $\iota(I(x)) = I(x)$, so we cannot have $\iota(x) < x$ – that would contradict the injectivity of ι – so it must be the case that $\iota(x) > x$. But since $\iota(X)$ is an initial segment, this means that x is in the image of ι, which is seen to be impossible. Now if $\alpha < \beta$ and $\beta < \alpha$ then we have initial embeddings $i : X \to Y$ and $j : Y \to X$. By Exercise 1.3.2 their composite $j \circ i : X \to X$ is an initial embedding, which we have just seen is impossible.

It remains to show that if $\alpha \neq \beta$ there is either initial embedding from X to Y or vice versa. We may assume that X is nonempty. Let us try to

Notes

build an initial embedding from X into Y . A little thought convinces us that we have no choices to make: suppose we have already defined an initial embedding f on a segment I(s) of X. Then we must define f(s) to be the least element of Y \ f(I(s)), and we can define it this way exactly when $f(I(s)) \neq Y$. If however f(I(s)) = Y , then we see that f^{-1} gives an initial embedding from Y to X. So assume Y is not isomorphic to an initial segment of X, and let Z be the set of x in X such that there exists an initial embedding from I(z) to Y . It is immediate to see that Z is an order ideal, so by Lemma 7 we have either Z = I(x) or Z = X. In the former case we have an initial embedding from I(z) to Y , and as above, the only we could not extend it to x is if it is surjective, and then we are done as above. So we can extend the initial embedding to I[x], which – again by Lemma 7 is either an initial segment (in which case we have a contradiction) or I[x] = X, in which case we are done. The last case is that Z = X has no maximal element, but then we have $X = \cup_{x \in X} I(x)$ and a uniquely defined initial embedding ι on each I(x). So altogether we have a map on all of X whose image f(X), as a union of initial segments, is an order ideal. Applying Lemma 7 yet again, we either have f(X) = Y – in which case f is an order isomorphism – or f(X) is an initial segment of Y , in which case X < Y : done.

Check In Progress-II

Q. 1 State Principal Ideal Domain

Solution :

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

Q. 2 Define Odering Odernilities.

Solution :

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

…………………………………………………………………………………

2.6 ZORN'S LEMMA

Introduction: Zorn's lemma is a result in set theory that appears in proofs of some non-constructive existence theorems throughout mathematics. We will state Zorn's lemma below and use it in later sections to prove some results in linear algebra, ring theory, group theory, and topology. In an appendix, we will give an application to metric spaces. The statement of Zorn's lemma is not intuitive, and some of the terminology in it may be unfamiliar, but after reading through the explanation of Zorn's lemma and then the proofs that use it you should be more comfortable with how it can be applied.

Theorem 1.1 (Zorn's lemma). Let S be a partially ordered set. If every totally ordered subset of S has an upper bound, then S contains a maximal element.

To understand Theorem 1.1, we need to know four terms: partially ordered set, totally ordered subset, upper bound, and maximal element. A partial ordering on a (nonempty) set S is a binary relation on S, denoted ≤, which satisfies the following properties:

- for all s ∈ S, s ≤ s,
- if s ≤ s 0 and s 0 ≤ s then s = s 0 ,
- if s ≤ s 0 and s 0 ≤ s 00 then s ≤ s 00 .

When we fix a partial ordering ≤ on S, we refer to S (or, more precisely, to the pair (S, ≤)) as a partially ordered set. It is important to notice that we do not assume all pairs of elements in S are comparable under ≤: for some s and s 0 we may have neither s ≤ s 0 nor s 0 ≤ s. If all pairs of elements can be compared (that is, for all s and s 0 in S either s ≤ s 0 or s 0 ≤ s) then we say S is totally ordered with respect to ≤.

Example 6.1. The usual ordering relation ≤ on R or on Z + is a partial ordering of these sets. In fact it is a total ordering on either set. This ordering on Z + is the basis for proofs by induction.

Example 6.2. On Z +, declare a ≤ b if a | b. This partial ordering on Z + is different from the one in Example 1.2 and is called ordering by divisibility. It is one of the central relations in number theory. (Proofs about Z + in number theory sometimes work not by induction, but by starting on primes, then extending to prime powers, and then extending to all positive integers using prime factorization. Such proofs view Z + through the divisibility relation rather than through the usual ordering relation.) Unlike the ordering on Z + in

Example 6.3, Z + is not totally ordered by divisibility: most pairs of integers are not comparable under the divisibility relation. For instance, 3 doesn't divide 5 and 5 doesn't divide 3. The subset {1, 2, 4, 8, 16, . . . } of powers of 2 is totally ordered under divisibility.

Example 6.4. Let S be the set of all subgroups of a given group G. For H, K ∈ S (that is, H and K are subgroups of G), declare H ≤ K if H is a subset of K. This is a partial ordering, called ordering by inclusion. It is not a total ordering: for most subgroups H and K neither H ⊂ K nor K ⊂ H. One can similarly partially order the subspaces of a vector space or the ideals (or subrings or all subsets) of a commutative ring by inclusion.

Example 6.5. On Z +, declare a ≤ b if b | a. Here one positive integer is "larger" than another if it is a factor. This is called ordering by reverse divisibility.

Example 6.6. On the set of subgroups of a group G, declare subgroups H and K to satisfy H ≤ K if K ⊂ H. This is a partial ordering on the subgroups of G, called ordering by reverse inclusion.

In case you think ordering by reverse inclusion seems weird, let's take a look again at Example 1.3. There positive integers are ordered by divisibility, and nothing seems "backwards." But let's associate to each a $\in$ Z + the subgroup aZ of Z. Every nonzero subgroup of Z has the form aZ for a unique positive integer a, aZ = bZ if and only if a = b (both a and b are positive), and a | b if and only if bZ $\subset$ aZ. For instance, 4 | 12 and 12Z $\subset$ 4Z. Therefore the ordering by divisibility on Z + is essentially the same as ordering by reverse inclusion on nonzero subgroups of Z. Partial ordering by reverse inclusion is used in the construction of completions of groups and rings.

Example 6.7. Let A and B be sets. Let S be the set of functions defined on some subset of A with values in B. The subset can vary with the function. That is, S is the set of pairs (X, f) where X $\subset$ A and f : X $\to$ B. Two elements (X, f) and (Y, g) in S are equal when X = Y and f(x) = g(x) for all x $\in$ X.

We can partially order S by declaring (X, f) $\leq$ (Y, g) when X $\subset$ Y and g|X = f. This means g is an extension of f to a larger subset of A. Let's check the second property of a partial ordering: if (X, f) $\leq$ (Y, g) and (Y, g) $\leq$ (X, f) then X $\subset$ Y and Y $\subset$ X, so X = Y . Then the condition g|X = f means g = f as functions on their common domain, so (X, f) = (Y, g).

Example 6.8. If S is a partially ordered set for the relation $\leq$ and T $\subset$ S, then the relation $\leq$ provides a partial ordering on T. Thus T is a new partially ordered set under $\leq$. For instance, the partial ordering by inclusion on the subgroups of a group restricts to a partial ordering on the cyclic subgroups of a group.

In these examples, only Example 1.2 is totally ordered. This is typical: most naturally occurring partial orderings are not total orderings. However (and this is important) a partially ordered set can have many subsets that are totally ordered. As a dumb example, every one-element subset of a partially ordered set is totally ordered. A more interesting illustration was at the end of Example 1.3 with the powers of 2 inside Z + under divisibility. As another example, if we partially order the subspaces of a vector space V by inclusion then any tower of subspaces

W1 $\subset$ W2 $\subset$ W3 $\subset \cdots$

where each subspace is a proper subset of the next one is a totally ordered subset of V . Here is a result about totally ordered subsets that will be useful at a few points later.

Lemma 1.9. Let S be a partially ordered set. If {s1, . . . , sn} is a finite totally ordered subset of S then there is an si such that sj ≤ si for all j = 1, . . . , n.

Proof. The si 's are all comparable to each other; that's what being totally ordered means. Since we're dealing with a finite set of pairwise comparable elements, there will be one that is greater than or equal to them all in the partial ordering on S. The reader can formalize this with a proof by induction on n, or think about the bubble sort algorithm

An upper bound on a subset T of a partially ordered set S is an s ∈ S such that t ≤ s for all t ∈ T. When we say T has an upper bound in S, we do not assume the upper bound is in T itself; it is just in S.

Example 6.10. In R with its natural ordering, the subset Z has no upper bound while the subset of negative real numbers has the upper bound 0 (or any positive real). No upper bound on the negative real numbers is a negative real number.

Example 6.11. In the proper subgroups of Z ordered by inclusion, an upper bound on {4Z, 6Z, 8Z} is 2Z since 4Z, 6Z, and 8Z all consist entirely of even numbers. (Note 4Z ⊂ 2Z, not 2Z ⊂ 4Z.)

A maximal element m of a partially ordered set S is an element that is not below any element to which it is comparable: for all s ∈ S to which m is comparable, s ≤ m. Equivalently, m is maximal when the only s ∈ S satisfying m ≤ s is s = m. This does not mean s ≤ m for all s in S since we don't insist that maximal elements are actually comparable to every element of S. A partially ordered set could have many maximal elements.

Example 6.12. If we partially order Z + by reverse divisibility (so a ≤ b means b | a), the number 1 is a maximal element. In fact 1 is the only maximal element. This is not a good example because 1 is comparable to everything in this relation, which is not a typical feature of maximal elements.

Example 6.13. Consider the positive integers greater than 1 with the reverse divisibility ordering: $a \leq b$ when $b \mid a$. The maximal elements here are the positive integers with no positive factor greater than 1 except themselves. These are the prime numbers, so the primes are the maximal elements for the reverse divisibility relation on $\{2, 3, 4, 5, 6, \ldots\}$

Equivalently, if we partially order the proper subgroups of Z by inclusion then the maximal elements are pZ for prime numbers p.

We now return to the statement of Zorn's lemma:

If every totally ordered subset of a partially ordered set S has an upper bound, then S contains a maximal element.

All the terms being used here have now been defined.1 Of course this doesn't mean the statement should be any clearer!

Zorn's lemma is not intuitive, but it turns out to be logically equivalent to more readily appreciated statements from set theory like the Axiom of Choice (which says the Cartesian product of any family of nonempty sets is nonempty). In the set theory appendix to [13], Zorn's lemma is derived from the Axiom of Choice. A proof of the equivalence between Zorn's lemma and the Axiom of Choice is given in the appendix to [16]. The reason for calling Zorn's lemma a lemma rather than an axiom is purely historical. Zorn's lemma is 1The hypotheses refer to all totally ordered subsets, and a totally ordered subset might be uncountable. Therefore it is a bad idea to write about "totally ordered sequences," since the label "sequence" is often understood to refer to a countably indexed set. Just use the label "totally ordered subset." that means a total ordering in which every nonempty subset has a least element), but do not confuse the totally ordered subsets in the hypotheses of Zorn's lemma with wellorderings on the whole set. They are different concepts, and you should never invoke the Well-Ordering Principle in the middle of an application of Zorn's lemma unless you really want to make bad mistakes.

Zorn's lemma provides no mechanism to find a maximal element whose existence it asserts. It also says nothing about how many maximal elements there are. Usually, as in Example 1.13, there are many maximal elements.

In a partially ordered set S we can speak about minimal elements just as much as maximal elements: m ∈ S is called minimal if m ≤ s for all s ∈ S to which m is comparable. Zorn's lemma can be stated in terms of minimal elements: if any totally ordered subset of a partially ordered set S has a lower bound in S then S has a minimal element. There really is no need to use this formulation, in practice, since by reversing the meaning of the partial ordering (that is, using the reverse ordering) lower bounds become upper bounds and minimal elements become maximal elements.

The applications of Zorn's lemma here are mostly to algebra, but it shows up in many other areas. For instance, the most important result in functional analysis is the HahnBanach theorem, whose proof uses Zorn's lemma. Another result from functional analysis, the Krein-Milman theorem, is proved using Zorn's lemma. (The Krein-Milman theorem is an example where Zorn's lemma is used to prove the existence of something that is more naturally a minimal element than a maximal element.) In topology, the most important theorem about compact spaces is Tychonoff's theorem, and it is proved using Zorn's lemma. When dealing with objects that have a built-in finiteness condition (such as finitedimensional vector spaces or finite products of spaces X1 × · · · × Xn), Zorn's lemma can be avoided by using ordinary induction in a suitable way (e.g., inducting on the dimension of a vector space). The essential uses of Zorn's lemma are for truly infinite objects, where one has to make infinitely many choices at once in a rather extreme way.

2.7 SUMMARY

We study in this unit well ordering set and Ordinal Trichotomy. We study well ordering set and its proposition with some examples. We study Partial Order set also.

2.8 KEYWORD

Ordinal trichotomy : A total order (or "totally ordered set," or "linearly ordered set") is a set plus a relation on the set (called a total order) that satisfies the conditions for a partial order plus an additional condition known as the comparability condition. ... Comparability

Well-Ordering : A well-order (or well-ordering or well-order relation) on a set S is a total order on S with the property that every non-empty subset of S has a least element in this ordering. The set S together with the well-order relation is then called a well-ordered set

Antidictionary : The set of all words of minimal length that never appear in a particular

2.9 EXERCISE

Exercise 1: Give an example of an order preserving bijection f such that f −1 is not order preserving.

Exercise 2. Prove Lemma 1. Lemma 2. (Rigidity Lemma) Let S and T be well-ordered sets and f1, f2 : S → T two order isomorphisms. Then f1 = f2. Proof: Let f1 and f2 be two order isomorphisms between the well-ordered sets S and T, which we may certainly assume are nonempty. Consider S2, the set of elements s of S such that f1(s)≠ f2(s), and let S1 = S \ S2. Since the least element of S must get mapped to the least element of T by any order-preserving map, S1 is nonempty; put T1 = f1(S1) = f2(S1). Supposing that S2 is nonempty, let s2 be its least element. Then f1(s2) and f2(s2) are both characterized by being the least element of T \ T1, so they must be equal, a contradiction.

Example 3: Trivially the empty set is well-ordered, as is any set of cardinality one. These sets, and only these sets, have unique well-orderings.

Exercise 4: Let α and β be ordinalities.

a) Show that 0β = 0, 1β = 1, α 0 = 1, α 1 = α.

b) Show that the correspondence between finite ordinals and natural numbers respects exponentiation.

c) For an ordinal α, the symbol α n now has two possible meanings: exponentiation and iterated multiplication. Show that the two ordinalities are equal. (The proof requires you to surmount a small left-to-right lexicographic difficulty.) In particular |α n| = |α| n = |α|.

d) For any infinite ordinal β, show that |α β | = max(|α|, |β|)

2.10 ANSWER FOR CHECK IN PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 3

Q 2 Check in Section 4

Check in Progress-II

Answer Q. 1 Check in Lemma 7

Q 2 Check in Section 4

2.11 SUGGESTION READING AND REFERENCES

[1] B. Banaschewski, A new proof that "Krull implies Zorn", Math. Logic Quart. 40 (1994), 478–480.

[2] J. L. Bell and F. Jellett, On the relationship between the Boolean prime ideal theorem and two principles in functional analysis, Bull. Acad. Polon. Sci. S´er. Sci. Math. Astronom. Phys. 19 (1971), 191–194.

[3] J. Bell and D. H. Fremlin, A geometric form of the axiom of choice, Fund. Math. 77 (1972), 167–170.

[4] A. Blass, Injectivity, projectivity, and the axiom of choice, Trans. Amer. Math. Soc. 255 (1979), 31–59.

[5] A. Blass, Existence of bases implies the axiom of choice, pp. 31–33 in: "Axiomatic set theory (Boulder, Colo., 1983)", Amer. Math. Soc., Providence, 1984.

[6] L. Blumenthal, "Theory and Applications of Distance Geometry," Clarendon Press, Oxford, 1953.

[7] D. Dummit and R. Foote, "Abstract Algebra," 3rd ed., Wiley, New York, 2004.

[8] J. D. Halpern and A. L´evy, The Boolean prime ideal theorem does not imply the axiom of choice, pp. 83–134 in: "Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I), Amer. Math. Soc., Providence, 1971.

[9] H. Herrlich, "Axiom of Choice," Springer-Verlag, Berlin, 2006.

[10] W. Hodges, Krull implies Zorn, J. London Math. Soc. 19 (1979), 285–287.

[11] J. Kelley, The Tychonoff product theorem implies the axiom of choice, Fund. Math. 37 (1950), 75–76.

[12] K. Keremedis, Bases for vector spaces over the two-element field and the axiom of choice, Proc. Amer. Math. Soc. 124 (1996), 2527–2531.

[13] S. Lang, "Algebra," 3rd revised ed., Springer, New York, 2002.

UNIT 3 ORDINAL AND CARDINAL NUMBERS

STRUCTURE

3.0 OBJECTIVE

* Here we learn cardinal number system
* after learning this unit we are able to know cardinal arithemetic
* learn successor of cardinal
* Learn cardinal arithemetic
* learn cardinal subtraction, multiplication, division and addition

3.1 INTRODUCTION

1 Introduction: A Cardinal Number is a number that says **how many** of something there are, such as one, two, three, four, five.

An Ordinal Number is a number that tells the **position** of something in a list, such as 1st, 2nd, 3rd, 4th, 5th etc.

Most ordinal numbers end in "th" except for:

1. one ⇒ first (1st)
2. two ⇒ second (2nd)
3. three ⇒ third (3rd)

Cardinal		Ordinal	
1	One	1st	First
2	Two	2nd	Second

Notes

3	Three	3rd	Third
4	Four	4th	Fourth
5	Five	5th	Fifth
6	Six	6th	Sixth
7	Seven	7th	Seventh
8	Eight	8th	Eighth
9	Nine	9th	Ninth
10	Ten	10th	Tenth
11	Eleven	11th	Eleventh
12	Twelve	12th	Twelfth
13	Thirteen	13th	Thirteenth
14	Fourteen	14th	Fourteenth
15	Fifteen	15th	Fifteenth
16	Sixteen	16th	Sixteenth
17	Seventeen	17th	Seventeenth
18	Eighteen	18th	Eighteenth
19	Nineteen	19th	Nineteenth
20	Twenty	20th	Twentieth
21	Twenty one	21st	Twenty-first
22	Twenty two	22nd	Twenty-second
23	Twenty three	23rd	Twenty-third
24	Twenty four	24th	Twenty-fourth
25	Twenty five	25th	Twenty-fifth
…	…	…	…
30	Thirty	30th	Thirtieth
31	Thirty one	31st	Thirty-first

32	Thirty two	32nd	Thirty-second
33	Thirty three	33rd	Thirty-third
34	Thirty four	34th	Thirty-fourth
…	…	…	…
40	Forty	40th	Fortieth
50	Fifty	50th	Fiftieth
60	Sixty	60th	Sixtieth
70	Seventy	70th	Seventieth
80	Eighty	80th	Eightieth
90	Ninety	90th	Ninetieth
100	One hundred	100th	Hundredth
…	…	…	…
1000	One thousand	1000th	Thousandth

3.2 CARDINAL NUMBERS

Cardinals for short, are a generalization of the natural numbers used to measure the cardinality (size) of sets. The cardinality of a finite set is a natural number: the number of elements in the set. The *transfinite* cardinal numbers describe the sizes of infinite sets.

Cardinality is defined in terms of bijective functions. Two sets have the same cardinality if, and only if, there is a one-to-one correspondence (bijection) between the elements of the two sets. In the case of finite sets, this agrees with the intuitive notion of size. In the case of infinite sets, the behavior is more complex. A fundamental theorem due to Georg Cantorshows that it is possible for infinite sets to have different cardinalities, and in particular the cardinality of the set of real numbers is greater than the cardinality of the set of natural numbers. It is also possible for a proper subset of an infinite set to have the same cardinality as the original set, something that cannot happen with proper subsets of finite sets.

This sequence starts with the natural numbers including zero (finite cardinals), which are followed by the aleph numbers (infinite cardinals of well-ordered sets). The aleph numbers are indexed by ordinal numbers. Under the assumption of the axiom of choice, this transfinite sequence includes every cardinal number. If one rejects that axiom, the situation is more complicated, with additional infinite cardinals that are not alephs.

Cardinality is studied for its own sake as part of set theory. It is also a tool used in branches of mathematics including model theory, combinatorics, abstract algebra, and mathematical analysis. In category theory, the cardinal numbers form a skeleton of the category of sets.

3.2.1 Motivation

In informal use, a **cardinal number** is what is normally referred to as a *counting number*, provided that 0 is included: 0, 1, 2, ... They may be identified with the natural numbers beginning with 0. The counting numbers are exactly what can be defined formally as the finite cardinal numbers. Infinite cardinals only occur in higher-level mathematics and logic.

More formally, a non-zero number can be used for two purposes: to describe the size of a set, or to describe the position of an element in a sequence. For finite sets and sequences it is easy to see that these two notions coincide, since for every number describing a position in a sequence we can construct a set which has exactly the right size, e.g. 3 describes the position of 'c' in the sequence <'a','b','c','d',...>, and we can construct the set {a,b,c} which has 3 elements. However, when dealing with infinite sets it is essential to distinguish between the two — the two notions are in fact different for infinite sets. Considering the position aspect leads to ordinal numbers, while the size aspect is generalized by the **cardinal numbers** described here.

The intuition behind the formal definition of cardinal is the construction of a notion of the relative size or "bigness" of a set without reference to the kind of members which it has. For finite sets this is easy; one simply counts the number of elements a set has. In order to compare the sizes of larger sets, it is necessary to appeal to more subtle notions.

A set *Y* is at least as big as a set *X* if there is an injective mapping from the elements of *X* to the elements of *Y*. An injective mapping identifies each element of the set *X* with a unique element of the set *Y*. This is most easily understood by an example; suppose we have the sets *X* = {1,2,3} and *Y* = {a,b,c,d}, then using this notion of size we would observe that there is a mapping:

1 → a

2 → b

3 → c

which is injective, and hence conclude that *Y* has cardinality greater than or equal to *X*. Note the element d has no element mapping to it, but this is permitted as we only require an injective mapping, and not necessarily an injective and onto mapping. The advantage of this notion is that it can be extended to infinite sets.

We can then extend this to an equality-style relation. Two sets *X* and *Y* are said to have the same **cardinality** if there exists a bijection between *X* and *Y*. By the Schroeder–Bernstein theorem, this is equivalent to there being *both* an injective mapping from *X* to *Y* *and* an injective mapping from *Y* to *X*. We then write $|X| = |Y|$. The cardinal number of *X* itself is often defined as the least ordinal *a* with $|a| = |X|$. This is called the von Neumann cardinal assignment; for this definition to make sense, it must be proved that every set has the same cardinality as *some* ordinal; this statement is the well-ordering principle. It is however possible to discuss the relative cardinality of sets without explicitly assigning names to objects.

The classic example used is that of the infinite hotel paradox, also called Hilbert's paradox of the Grand Hotel. Suppose you are an innkeeper at a hotel with an infinite number of rooms. The hotel is full, and then a new guest arrives. It is possible to fit the extra guest in by asking the guest who was in room 1 to move to room 2, the guest in room 2 to move to room 3, and so on, leaving room 1 vacant. We can explicitly write a segment of this mapping:

$1 \to 2$

$2 \to 3$

$3 \to 4$

...

$n \to n + 1$

...

In this way we can see that the set {1,2,3,...} has the same cardinality as the set {2,3,4,...} since a bijection between the first and the second has been shown. This motivates the definition of an infinite set being any set which has a proper subset of the same cardinality; in this case {2,3,4,...} is a proper subset of {1,2,3,...}.

When considering these large objects, we might also want to see if the notion of counting order coincides with that of cardinal defined above for these infinite sets. It happens that it doesn't; by considering the above example we can see that if some object "one greater than infinity" exists, then it must have the same cardinality as the infinite set we started out with. It is possible to use a different formal notion for number, called ordinals, based on the ideas of counting and considering each number in turn, and we discover that the notions of cardinality and ordinality are divergent once we move out of the finite numbers.

It can be proved that the cardinality of the real numbers is greater than that of the natural numbers just described. This can be visualized using Cantor's diagonal argument; classic questions of cardinality (for instance the continuum hypothesis) are concerned with discovering whether there is some cardinal between some pair of other infinite cardinals. In more recent times mathematicians have been describing the properties of larger and larger cardinals.

Since cardinality is such a common concept in mathematics, a variety of names are in use. Sameness of cardinality is sometimes referred to as **equipotence**, **equipollence**, or **equinumerosity**. It is thus said that two sets with the same cardinality are, respectively, **equipotent**, **equipollent**, or **equinumerous**.

3.2.2 Formal Definition

Formally, assuming the axiom of choice, the cardinality of a set *X* is the least ordinal number α such that there is a bijection between *X* and α. This definition is known as the von Neumann cardinal assignment. If the axiom of choice is not assumed we need to do something different. The oldest definition of the cardinality of a set *X* (implicit in Cantor and explicit in Frege and Principia Mathematica) is as the class *[X]* of all sets that are equinumerous with *X*. This does not work in ZFC or other related systems of axiomatic set theory because if *X* is non-empty, this collection is too large to be a set. In fact, for $X \neq \emptyset$ there is an injection from the universe into *[X]* by mapping a set *m* to $\{m\} \times X$ and so by the axiom of limitation of size, *[X]* is a proper class. The definition does work however in type theory and in New Foundations and related systems. However, if we restrict from this class to those equinumerous with *X* that have the least rank, then it will work (this is a trick due to Dana Scott:[2] it works because the collection of objects with any given rank is a set).

Formally, the order among cardinal numbers is defined as follows: $|X| \leq |Y|$ means that there exists an injective function from *X* to *Y*. The Cantor–Bernstein–Schroeder theorem states that if $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$. The axiom of choice is equivalent to the statement that given two sets *X* and *Y*, either $|X| \leq |Y|$ or $|Y| \leq |X|$.

A set *X* is Dedekind-infinite if there exists a proper subset *Y* of *X* with $|X| = |Y|$, and Dedekind-finite if such a subset doesn't exist. The finite cardinals are just the natural numbers, i.e., a set *X* is finite if and only if $|X| = |n| = n$ for some natural number *n*. Any other set is infinite. Assuming the axiom of choice, it can be proved that the Dedekind notions correspond to the standard ones. It can also be proved that the cardinal (aleph null or aleph-0, where aleph is the first letter in the Hebrew alphabet, represented) of the set of natural numbers is the smallest infinite cardinal, i.e. that any infinite set has a subset of cardinality The next larger cardinal is denoted by and so on. For every ordinal α there is a cardinal number and this list exhausts all infinite cardinal numbers.

Check in Progress-I

Q. 1 Define Cardinal Numbers.

Solution. …………………………………………………………………..

…………………………………………………………………………….

……………………………………………………………………………

…………………………………………………………………………..

Q. 2 What is Cardinal Numbers.

Solution. …………………………………………………………………..

…………………………………………………………………………….

……………………………………………………………………………

…………………………………………………………………………..

3.3 CARDINAL ARITHMETIC

We can define arithmetic operations on cardinal numbers that generalize the ordinary operations for natural numbers. It can be shown that for finite cardinals these operations coincide with the usual operations for natural numbers. Furthermore, these operations share many properties with ordinary arithmetic.

3.3.1 Successor Cardinal

If the axiom of choice holds, then every cardinal κ has a successor $\kappa^+ > \kappa$, and there are no cardinals between κ and its successor. (Without the axiom of choice, using Hartogs' theorem, it can be shown that, for any cardinal number κ, there is a minimal cardinal κ^+, such that) For finite cardinals, the successor is simply κ + 1. For infinite cardinals, the successor cardinal differs from the successor ordinal.

3.3.2 Cardinal Addition

If *X* and *Y* are disjoint, addition is given by the union of *X* and *Y*. If the two sets are not already disjoint, then they can be replaced by disjoint sets of the same cardinality, e.g., replace *X* by $X\times\{0\}$ and *Y* by $Y\times\{1\}$.

Zero is an additive identity $\kappa + 0 = 0 + \kappa = \kappa$.

Addition is associative $(\kappa + \mu) + \nu = \kappa + (\mu + \nu)$.

Addition is commutative $\kappa + \mu = \mu + \kappa$.

Addition is non-decreasing in both arguments:

Assuming the axiom of choice, addition of infinite cardinal numbers is

easy. If either κ or μ is infinite, then

3.3.3 Cardinal Subtraction

Assuming the axiom of choice and, given an infinite cardinal σ and a cardinal μ, there exists a cardinal κ such that $\mu + \kappa = \sigma$ if and only if $\mu \le \sigma$. It will be unique (and equal to σ) if and only if $\mu < \sigma$.

3.3.4 Cardinal Multiplication

The product of cardinals comes from the cartesian product.

$\kappa\cdot 0 = 0\cdot\kappa = 0$.

$\kappa\cdot\mu = 0 \rightarrow (\kappa = 0 \text{ or } \mu = 0)$.

One is a multiplicative identity $\kappa\cdot 1 = 1\cdot\kappa = \kappa$.

Multiplication is associative $(\kappa\cdot\mu)\cdot\nu = \kappa\cdot(\mu\cdot\nu)$.

Multiplication is commutative $\kappa\cdot\mu = \mu\cdot\kappa$.

Multiplication is non-decreasing in both arguments: $\kappa \le \mu \rightarrow (\kappa\cdot\nu \le \mu\cdot\nu$ and $\nu\cdot\kappa \le \nu\cdot\mu)$.

Notes

Multiplication distributes over addition: $\kappa\cdot(\mu + \nu) = \kappa\cdot\mu + \kappa\cdot\nu$ and $(\mu + \nu)\cdot\kappa = \mu\cdot\kappa + \nu\cdot\kappa$.

Assuming the axiom of choice, multiplication of infinite cardinal numbers is also easy. If either κ or μ is infinite and both are non-zero, then

3.3.5 Cardinal Division

Assuming the axiom of choice and, given an infinite cardinal π and a non-zero cardinal μ, there exists a cardinal κ such that $\mu \cdot \kappa = \pi$ if and only if $\mu \leq \pi$. It wbe unique (and equal to π) if and only if $\mu < \pi$.

3.3.6 Cardinal Exponentiation

Exponentiation is given by

where X^Y is the set of all functions from Y to X.

$\kappa^0 = 1$ (in particular $0^0 = 1$), see empty function.

If $1 \leq \mu$, then $0^\mu = 0$.

$1^\mu = 1$.

$\kappa^1 = \kappa$.

$\kappa^{\mu + \nu} = \kappa^\mu\cdot\kappa^\nu$.

$\kappa^{\mu \cdot \nu} = (\kappa^\mu)^\nu$.

$(\kappa\cdot\mu)^\nu = \kappa^\nu\cdot\mu^\nu$.

Exponentiation is non-decreasing in both arguments:

$(1 \leq \nu$ and $\kappa \leq \mu) \rightarrow (\nu^\kappa \leq \nu^\mu)$ and

$(\kappa \leq \mu) \rightarrow (\kappa^\nu \leq \mu^\nu)$.

Note that $2^{|X|}$ is the cardinality of the power set of the set X and Cantor's diagonal argument shows that $2^{|X|} > |X|$ for any set X. This proves that no largest cardinal exists (because for any cardinal κ, we can always find a larger cardinal 2^κ). In fact, the class of cardinals is a proper class. (This proof fails in some set theories, notably New Foundations.)

All the remaining propositions in this section assume the axiom of choice:

If κ and μ are both finite and greater than 1, and ν is infinite, then $\kappa^{\nu} = \mu^{\nu}$.

If κ is infinite and μ is finite and non-zero, then $\kappa^{\mu} = \kappa$.

If $2 \leq \kappa$ and $1 \leq \mu$ and at least one of them is infinite, then:

$\text{Max}(\kappa, 2^{\mu}) \leq \kappa^{\mu} \leq \text{Max}(2^{\kappa}, 2^{\mu})$.

Using König's theorem, one can prove $\kappa < \kappa^{cf(\kappa)}$ and $\kappa < cf(2^{\kappa})$ for any infinite cardinal κ, where cf(κ) is the cofinality of κ.

3.3.7 Cardinal Roots

Assuming the axiom of choice and, given an infinite cardinal κ and a finite cardinal μ greater than 0, the cardinal ν satisfying will be κ.

3.3.8 Cardinal Logarithms

Assuming the axiom of choice and, given an infinite cardinal κ and a finite cardinal μ greater than 1, there may or may not be a cardinal λ satisfying . However, if such a cardinal exists, it is infinite and less than κ, and any finite cardinality ν greater than 1 will also satisfy .

The logarithm of an infinite cardinal number κ is defined as the least crdinal number μ such that $\kappa \leq 2^{\mu}$. Logarithms of infinite cardinals are useful in some fields of mathematics, for example in the studyof cardinalvariants of topological spaces, though they lack some of the properties that logarithms of positive real numbers possess.[5][6][7]

The continuum hypothesis

The continuum hypothesis (CH) states that there are no cardinals strictly The latter cardinal number is also often denoted by ; it is the cardinality of the continuum (the set of real numbers). In this case The generalized continuum hypothesis (GCH) states that for every infinite set X, there are no cardinals strictly between $|X|$ and $2^{|X|}$. The continuum hypothesis is independent of the usual axioms of set theory, the Zermelo-Fraenkel axioms together with the axiom of choice (ZFC).

3.4. WELL-ORDERING PRINCIPLE

In mathematics, the **well-ordering principle** states that every non-empty set of positive integers contains a least element.[1] In other words, the set of positive integers is well-ordered by its "natural" or "magnitude" order in which precedes if and only if is either or the sum of and some positive integer (other orderings include the ordering ; and).

The phrase "well-ordering principle" is sometimes taken to be synonymous with the "well-ordering theorem". On other occasions it is understood to be the proposition that the set of integers contains a well-ordered subset, called the natural numbers, in which every nonempty subset contains a least element.

Depending on the framework in which the natural numbers are introduced, this (second order) property of the set of natural numbers is either an axiom or a provable theorem. For example:

- In Peano arithmetic, second-order arithmetic and related systems, and indeed in most (not necessarily formal) mathematical treatments of the well-ordering principle, the principle is derived from the principle of mathematical induction, which is itself taken as basic.
- Considering the natural numbers as a subset of the real numbers, and assuming that we know already that the real numbers are complete (again, either as an axiom or a theorem about the real number system), i.e., every bounded (from below) set has an infimum, then also every set of natural numbers has an infimum, say . We can now find an integer such that lies in the half-open interval , and can then

  show that we must have , and in .
- In axiomatic set theory, the natural numbers are defined as the smallest inductive set (i.e., set containing 0 and closed under the successor operation). One can (even without invoking the regularity axiom) show that the set of all natural numbers such that " is well-ordered" is inductive, and must therefore contain all natural numbers; from this property one can conclude that the set of all natural numbers is also well-ordered.

In the second sense, this phrase is used when that proposition is relied on for the purpose of justifying proofs that take the following form: to prove that every natural number belongs to a specified set S , assume the contrary, which implies that the set of counterexamples is non-empty and thus contains a smallest counterexample. Then show that for any counterexample there is a still smaller counterexample, producing a contradiction. This mode of argument is the contrapositive of proof by complete induction. It is known light-heartedly as the "minimal criminal" method and is similar in its nature to Fermat's method of "infinite descent".

Garrett Birkhoff and Saunders Mac Lane wrote in *A Survey of Modern Algebra* that this property, like the least upper bound axiom for real numbers, is non-algebraic; i.e., it cannot be deduced from the algebraic properties of the integers (which form an ordered integral domain).

3.4.1 Well-ordered sets

In a well-ordered set, every non-empty subset contains a distinct smallest element. Given the axiom of dependent choice, this is equivalent to just saying that the set is totally ordered and there is no infinite decreasing sequence, something perhaps easier to visualize. In practice, the importance of well-ordering is justified by the possibility of applying transfinite induction, which says, essentially, that any property that passes on from the predecessors of an element to that element itself must be true of all elements (of the given well-ordered set). If the states of a computation (computer program or game) can be well-ordered in such a way that each step is followed by a "lower" step, then the computation will terminate.

It is inappropriate to distinguish between two well-ordered sets if they only differ in the "labeling of their elements", or more formally: if the elements of the first set can be paired off with the elements of the second set such that if one element is smaller than another in the first set, then the partner of the first element is smaller than the partner of the second element in the second set, and vice versa. Such a one-to-one correspondence is called an order isomorphism and the two well-ordered sets are said to be order-isomorphic, or *similar* (obviously this is an equivalence relation). Provided there exists an order isomorphism between two well-ordered sets, the order isomorphism is unique: this makes it quite justifiable to consider the two sets as essentially identical, and to seek a "canonical" representative of the isomorphism type (class). This is exactly what the ordinals provide, and it also provides a canonical labeling of the elements of any well-ordered set. Formally, if a partial order < is defined on the set S, and a partial order <' is defined on the set S', then the posets (S,<) and (S',<') are order isomorphic if there is a bijection f that preserves the ordering. That is, $f(a)$ <' $f(b)$ if and only if $a < b$. Every *well-ordered* set (S,<) is order isomorphic to the set of ordinals less than one specific ordinal number [the order type of (S,<)] under their natural ordering.

Essentially, an ordinal is intended to be defined as an isomorphism class of well-ordered sets: that is, as an equivalence class for the equivalence relation of "being order-isomorphic". There is a technical difficulty involved, however, in the fact that the equivalence class is too large to be a set in the usual Zermelo–Fraenkel (ZF) formalization of set theory. But this is not a serious difficulty. The ordinal can be said to be the *order type*of any set in the class.

3.5. ORDINAL NUMBER

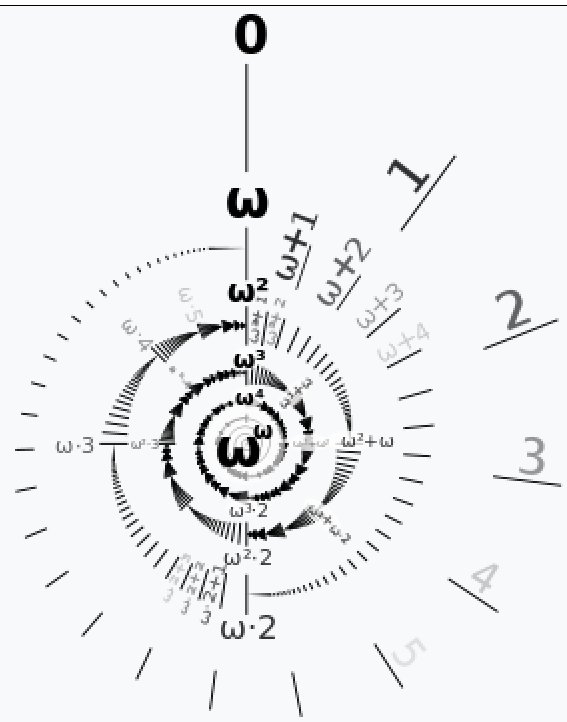


Figure 5.1

Representation of the ordinal numbers up to ω^{ω}. Each turn of the spiral represents one power of ω

In set theory, an **ordinal number**, or **ordinal**, is one generalization of the concept of a natural number that is used to describe a way to arrange a collection of objects in order, one after another. Any finite collection of objects can be put in order just by the process of counting: labeling the objects with distinct natural numbers. Ordinal numbers are thus the "labels" needed to arrange collections of objects in order.

An ordinal number is used to describe the order type of a well-ordered set (though this does not work for a well-ordered proper class). A well-ordered set is a set with a relation > such that

Trichotomy

For any elements x and y, exactly one of these statements is true

- $x > y$
- $y = x$
- $y > x$

Transitivity

For any elements x, y, z, if $x > y$ and $y > z$, then $x > z$

Well-foundedness

Every non-empty subset has a least element, that is, it has an element x such that there is no other element y in the subset where $x > y$

Two well-ordered sets have the same order type if and only if there is a bijection from one set to the other that converts the relation in the first set to the relation in the second set.

Whereas ordinals are useful for *ordering* the objects in a collection, they are distinct from cardinal numbers, which are useful for saying how many objects are in a collection. Although the distinction between ordinals and cardinals is not always apparent in finite sets (one can go from one to the other just by counting labels), different infinite ordinals can describe the same cardinal. Like other kinds of numbers, ordinals can be added, multiplied, and exponentiated, although the addition and multiplication are not commutative.

3.5.1 Von Neumann definition of ordinals

First few von Neumann ordinals	
0 = { }	= ∅
1 = { 0 }	= {∅}
2 = { 0, 1 }	= { ∅, {∅} }
3 = { 0, 1, 2 }	= { ∅, {∅} , {∅, {∅}} }
4 = { 0, 1, 2, 3 }	= { ∅, {∅}, {∅, {∅}}, {∅, {∅}, {∅, {∅}}} }

Rather than defining an ordinal as an *equivalence class* of well-ordered sets, it will be defined as a particular well-ordered set that (canonically) represents the class. Thus, an ordinal number will be a well-ordered set; and every well-ordered set will be order-isomorphic to exactly one ordinal number.

For each well-ordered set , defines an order isomorphism between and the set of all subsets of having the form ordered by inclusion. This motivates the standard definition, suggested by John von Neumann, now called definition of von Neumann ordinals: "each ordinal is the well-ordered set of all smaller ordinals." In symbols, λ = [0,λ).[3][4] Formally:

> A set *S* is an ordinal if and only if *S* is strictly well-ordered with respect to set membership and every element of *S* is also a subset of *S*.

The natural numbers are thus ordinals by this definition. For instance, 2 is an element of 4 = {0, 1, 2, 3}, and 2 is equal to {0, 1} and so it is a subset of {0, 1, 2, 3}.

It can be shown by transfinite induction that every well-ordered set is order-isomorphic to exactly one of these ordinals, that is, there is an order preserving bijective function between them.

Furthermore, the elements of every ordinal are ordinals themselves. Given two ordinals *S* and *T*, *S* is an element of *T* if and only if *S* is a proper subset of T. Moreover, either S is an element of T, or T is an element of S, or they are equal. So every set of ordinals is totally ordered. Further, every set of ordinals is well-ordered. This generalizes the fact that every set of natural numbers is well-ordered.

Consequently, every ordinal S is a set having as elements precisely the ordinals smaller than S. For example, every set of ordinals has a supremum, the ordinal obtained by taking the union of all the ordinals in the set. This union exists regardless of the set's size, by the axiom of union.

The class of all ordinals is not a set. If it were a set, one could show that it was an ordinal and thus a member of itself, which would contradict its *strict* ordering by membership. This is the Burali-Forti paradox. The class of all ordinals is variously called "Ord", "ON", or "∞".

An ordinal is finite if and only if the opposite order is also well-ordered, which is the case if and only if each of its subsets has a maximum.

3.5.2 Transfinite Sequence

If α is a limit ordinal and X is a set, an α-indexed sequence of elements of X is a function from α to X. This concept, a **transfinite sequence** or **ordinal-indexed sequence**, is a generalization of the concept of a sequence. An ordinary sequence corresponds to the case $\alpha = \omega$.

3.5.3 Transfinite Induction

Transfinite induction holds in any well-ordered set, but it is so important in relation to ordinals that it is worth restating here.

> Any property that passes from the set of ordinals smaller than a given ordinal α to α itself, is true of all ordinals.

That is, if $P(\alpha)$ is true whenever $P(\beta)$ is true for all $\beta < \alpha$, then $P(\alpha)$ is true for *all* α. Or, more practically: in order to prove a property P for all ordinals α, one can assume that it is already known for all smaller $\beta < \alpha$.

3.5.4 Successor and limit Ordinals

Any nonzero ordinal has the minimum element, zero. It may or may not have a maximum element. For example, 42 has maximum 41 and ω+6 has maximum ω+5. On the other hand, ω does not have a maximum since there is no largest natural number. If an ordinal has a maximum α, then it is the next ordinal after α, and it is called a *successor ordinal*, namely the successor of α, written α+1. In the von Neumann definition of ordinals, the successor of α is since its elements are those of α and α itself.[3]

A nonzero ordinal that is *not* a successor is called a *limit ordinal*. One justification for this term is that a limit ordinal is the limit in a topological sense of all smaller ordinals (under the order topology).

When is an ordinal-indexed sequence, indexed by a limit γ and the sequence is *increasing*, i.e. whenever its *limit* is defined as the least upper bound of the set that is, the smallest ordinal (it always exists) greater than any term of the sequence. In this sense, a limit ordinal is the limit of all smaller ordinals (indexed by itself). Put more directly, it is the supremum of the set of smaller ordinals.

Another way of defining a limit ordinal is to say that α is a limit ordinal if and only if:

> There is an ordinal less than α and whenever ζ is an ordinal less than α, then there exists an ordinal ξ such that $\zeta < \xi < \alpha$.

So in the following sequence:

0, 1, 2, …, ω, ω+1

ω is a limit ordinal because for any smaller ordinal (in this example, a natural number) there is another ordinal (natural number) larger than it, but still less than ω.

Thus, every ordinal is either zero, or a successor (of a well-defined predecessor), or a limit. This distinction is important, because many definitions by transfinite induction rely upon it.

Very often, when defining a function *F* by transfinite induction on all ordinals, one defines $F(0)$, and $F(\alpha+1)$ assuming $F(\alpha)$ is defined, and then, for limit ordinals δ one defines $F(\delta)$ as the limit of the $F(\beta)$ for all $\beta<\delta$ (either in the sense of ordinal limits, as previously explained, or for some other notion of limit if *F* does not take ordinal values). Thus, the interesting step in the definition is the successor step, not the limit ordinals. Such functions (especially for *F*nondecreasing and taking ordinal values) are called continuous. Ordinal addition, multiplication and exponentiation are continuous as functions of their second argument.

Check In Progress-II

Q. 1 Dfine Ordinal Number.

Solution :

……………………………………………………………………………..

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

Q. 2 Define Well Ordering Principal.

Solution :

……………………………………………………………………………..

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

Notes

3.5.5 Indexing Classes of Ordinal

Any well-ordered set is similar (order-isomorphic) to a unique ordinal number ; in other words, its elements can be indexed in increasing fashon by the ordinals less than . This applies, in particular, to any set of ordinals: any set of ordinals is naturally indexed by the ordinals less than some . The same holds, with a slight modification, for *classes* of ordinals (a collection of ordinals, possibly too large to form a set, defined by some property): any class of ordinals can be indexed by ordinals (and, when the class is unbounded in the class of all ordinals,

this puts it in class-bijection with the class of all ordinals). So the -th element in the class (with the convention that the "0-th" is the smallest, the "1-st" is the next smallest, and so on) can be freely spoken of. Formally, the definition is by transfinite induction: the -th element of the class is defined (provided it has already been defined for all), as the

smallest element greater than the -th element for all .

This could be applied, for example, to the class of limit ordinals: the -th ordinal, which is either a limit or zero is (see ordinal arithmetic for the definition of multiplication of ordinals). Similarly, one can consider *additively indecomposable ordinals* (meaning a nonzero ordinal that is not the sum of two strictly smaller ordinals): the -th additively indecomposable ordinal is indexed as S. The technique of indexing classes of ordinals is often useful in the context of fixed points:

3.5.6 Closed Unbounded Sets and Classes

A class of ordinals is said to be **unbounded**, or **cofinal**, when given any ordinal , there is a in such that (then the class must be a proper class, i.e., it cannot be a set). It is said to be **closed** when the limit of a sequence of ordinals in the class is again in the class: or, equivalently, when the indexing (class-)function is continuous in the sense that, for a limit ordinal, (the -th ordinal in the class) is the limit of all for ; this is also the same as being closed, in the topological sense, for the order topology (to avoid talking of topology on proper classes, one can demand that the intersection of the class with any given ordinal is closed for the order topology on that ordinal, this is again equivalent).

Of particular importance are those classes of ordinals that are closed and unbounded, sometimes called **clubs**. For example, the class of all limit ordinals is closed and unbounded: this translates the fact that there is always a limit ordinal greater than a given ordinal, and that a limit of limit ordinals is a limit ordinal (a fortunate fact if the terminology is to make any sense at all!). The class of additively indecomposable ordinals, or the class of ordinals, or the class of cardinals, are all closed unbounded; the set of regular cardinals, however, is unbounded but not closed, and any finite set of ordinals is closed but not unbounded.

A class is stationary if it has a nonempty intersection with every closed unbounded class. All superclasses of closed unbounded classes are stationary, and stationary classes are unbounded, but there are stationary classes that are not closed and stationary classes that have no closed unbounded subclass (such as the class of all limit ordinals with countable cofinality). Since the intersection of two closed unbounded classes is closed and unbounded, the intersection of a stationary class and a closed unbounded class is stationary. But the intersection of two stationary classes may be empty, e.g. the class of ordinals with cofinality ω with the class of ordinals with uncountable cofinality.

Rather than formulating these definitions for (proper) classes of ordinals, one can formulate them for sets of ordinals below a given ordinal : A subset of a limit ordinal is said to be unbounded (or cofinal) under provided any ordinal less than is less than some ordinal in the set. More generally, one can call a subset of any ordinal cofinal in provided every ordinal less than is less than *or equal to* some ordinal in the set. The subset is said to be closed under provided it is closed for the order topology *in* , i.e. a limit of ordinals in the set is either in the set or equal to itself.

3.6 ARITHMETIC OF ORDINALS

There are three usual operations on ordinals: addition, multiplication, and (ordinal) exponentiation. Each can be defined in essentially two different ways: either by constructing an explicit well-ordered set that represents the operation or by using transfinite recursion. The Cantor normal form provides a standardized way of writing ordinals. It uniquely represents each ordinal as a finite sum of ordinal powers of ω. However, this cannot form the basis of a universal ordinal notation due to such self-referential representations as $\varepsilon_0 = \omega^{\varepsilon_0}$. The so-called "natural" arithmetical operations retain commutativity at the expense of continuity.

Interpreted as nimbers, ordinals are also subject to nimber arithmetic operations.

3.6.1 Ordinals and cardinals

Initial ordinal of a cardinal

Each ordinal associates with one cardinal, its cardinality. If there is a bijection between two ordinals (e.g. $\omega = 1 + \omega$ and $\omega + 1 > \omega$), then they associate with the same cardinal. Any well-ordered set having an ordinal as its order-type has the same cardinality as that ordinal. The least ordinal associated with a given cardinal is called the *initial ordinal* of that cardinal. Every finite ordinal (natural number) is initial, and no other ordinal associates with its cardinal. But most infinite ordinals are not initial, as many infinite ordinals associate with the same cardinal. The axiom of choice is equivalent to the statement that every set can be well-ordered, i.e. that every cardinal has an initial ordinal. In theories with the axiom of choice, the cardinal number of any set has an initial ordinal, and one may employ the Von Neumann cardinal assignment as the cardinal's representation. In set theories without the axiom of choice, a cardinal may be represented by the set of sets with that cardinality having minimal rank (see Scott's trick).

The α-th infinite initial ordinal is written , it is always a limit ordinal. Its cardinality is written . For example, the cardinality of $\omega_0 = \omega$ is , which is also the cardinality of ω^2 or ε_0 (all are countable ordinals). So ω can be identified with , except that the notation is used when writing cardinals, and ω when writing ordinals (this is important since, for example, = whereas). Also, is the smallest uncountable ordinal (to see that it exists, consider the set of equivalence classes of well-orderings of the natural numbers: each such well-ordering defines a countable ordinal, and is the order type of that set), is the smallest ordinal whose cardinality is greater than , and so on, and is the limit of the for natural numbers *n* (any limit of cardinals is a cardinal, so this limit is indeed the first cardinal after all the).

3.6.2 Cofinality

The cofinality of an ordinal is the smallest ordinal that is the order type of a cofinal subset of . Notice that a number of authors define cofinality or use it only for limit ordinals. The cofinality of a set of ordinals or any other well-ordered set is the cofinality of the order type of that set.

Thus for a limit ordinal, there exists a -indexed strictly increasing sequence with limit . For example, the cofinality of ω^2 is ω, because the sequence $\omega \cdot m$ (where m ranges over the natural numbers) tends to ω^2; but, more generally, any countable limit ordinal has cofinality ω. An uncountable limit ordinal may have either cofinality ω as does or an uncountable cofinality.

The cofinality of 0 is 0. And the cofinality of any successor ordinal is 1.

The cofinality of any limit ordinal is at least .

An ordinal that is equal to its cofinality is called regular and it is always an initial ordinal. Any limit of regular ordinals is a limit of initial ordinals and thus is also initial even if it is not regular, which it usually is not. If the Axiom of Choice, then is regular for each α. In this case, the ordinals 0, 1, , , and are regular, whereas 2, 3, , and $\omega_{\omega \cdot 2}$ are initial ordinals that are not regular.

The cofinality of any ordinal α is a regular ordinal, i.e. the cofinality of the cofinality of α is the same as the cofinality of α. So the cofinality operation is idempotent.

3.6.3 Topology and Ordinals

Any ordinal number can be made into a topological space by endowing it with the order topology; this topology is discrete if and only if the ordinal is a countable cardinal, i.e. at most ω. A subset of $\omega + 1$ is open in the order topology if and only if either it is cofinite or it does not contain ω as an element.

See the Topology and ordinals section of the "Order topology" article.

3.6.4 Downward Closed Sets of Ordinals

A set is downward closed if anything less than an element of the set is also in the set. If a set of ordinals is downward closed, then that set is an ordinal—the least ordinal not in the set.

Examples:

- The set of ordinals less than 3 is 3 = { 0, 1, 2 }, the smallest ordinal not less than 3.
- The set of finite ordinals is infinite, the smallest infinite ordinal: ω.
- The set of countable ordinals is uncountable, the smallest uncountable ordinal: ω_1.

3.7 SUMMARY

We study in this unit about Cardinal numbers and its definition. We study Downward closed sets of Ordinals. We study Cofinality and its introduction. We study success of limit cardinal. We study Ordinal Numbers.

3.8 KEYWORD

CARDINAL: A leading dignitary of the Roman Catholic Church. Cardinals are nominated by the Pope, and form the Sacred College which elects succeeding popes (now invariably from among their own number

ORDINAL : A service book, especially one with the forms of service used at ordinations.

IDEMPOTENT : Denoting an element of a set which is unchanged in value when multiplied or otherwise operated on by itself

3.9 EXERCISE

Q. 1 What is cardinals numbers ? Give example.

Q. 2 What is Ordinal Number ? Give example.

Q. 3 What is well order set ?

Q. 4 In topology what is Idempotent ?

Q. 5 What is cardinal arithemetic ?

3.10 ANSWER FOR CHECK IN PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 3

Q 2 Check in Section 4

Check in Progress-II

Answer Q. 1 Check in Section 1

Q 2 Check in Section 1

3.11 SUGGESTION READING AND REFERENCES

1. Dauben 1990, pg. 54
2. ^ *Deiser, Oliver (May 2010). "On the Development of the Notion of a Cardinal Number". History and Philosophy of Logic. **31** (2): 123–143. doi:10.1080/01445340903545904.*
3. ^ Enderton, Herbert. "Elements of Set Theory", Academic Press Inc., 1977. ISBN 0-12-238440-7
4. ^ *Friedrich M. Hartogs (1915), Felix Klein; Walther von Dyck; David Hilbert; Otto Blumenthal (eds.), "Über das Problem der Wohlordnung", Math. Ann., Leipzig: B. G. Teubner, Bd. 76 (4): 438–443, doi:10.1007/bf01458215, ISSN 0025-5831*
5. ^ Robert A. McCoy and Ibula Ntantu, Topological Properties of Spaces of Continuous Functions, Lecture Notes in Mathematics 1315, Springer-Verlag.
6. ^ Eduard Čech, Topological Spaces, revised by Zdenek Frolík and Miroslav Katetov, John Wiley & Sons, 1966.

7. ^ D. A. Vladimirov, Boolean Algebras in Analysis, Mathematics and Its Applications, Kluwer Academic Publishers.

3.12 BIBLIOGRAPHY

- *Dauben, Joseph Warren (1990), Georg Cantor: His Mathematics and Philosophy of the Infinite, Princeton: Princeton University Press, ISBN 0691-02447-2*
- Hahn, Hans, *Infinity*, Part IX, Chapter 2, Volume 3 of *The World of Mathematics*. New York: Simon and Schuster, 1956.
- Halmos, Paul, *Naive set theory*. Princeton, NJ: D. Van Nostrand Company, 1960. Reprinted by Springer-Verlag, New York, 1974. ISBN 0-387-90092-6 (Springer-Verlag edition).

UNIT 4 TOPOLOGICAL SPACE

STRUCTURE

4.0 OBJECTIVE

* We study in this unit basis of topology

* learn subspace topology, closed set, limit point

* Subbase topology

* Learn Alexander subbase theorem

* learn order topology

4.1 INTRODUCTION

In topology and related branches of mathematics, a topological space may be defined as a set of points, along with a set of neighbourhoods for each point, satisfying a set

Notes

of axioms relating points and neighbourhoods. The definition of a topological space relies only upon set theory and is the most general notion of a mathematical space that allows for the definition of concepts such as continuity, connectedness, and convergence. Other spaces, such as manifolds and metric spaces, are specializations of topological spaces with extra structures or constraints. Being so general, topological spaces are a central unifying notion and appear in virtually every branch of modern mathematics. The branch of mathematics that studies topological spaces in their own right is called point-set topology or general topology.

Definition 0.1.1. A *topology* on a set X is a collection J of subsets of X having the following properties:

(i)$\emptyset$ and X are in J .

(ii) The union of the elements of any subcollection of J is in J .

(iii) The intersection of the elements of any finite subcollection of J is in J .

A set X for which a topology J has been specified is called a *topological space*. If X is a topological space with topology J , we say that a subset U of X is an *open set* of X , if U belongs to the collection J .

If X is any set, the collection of all subsets of X is a topology on X , it is called the *discrete topology*. The collection consisting of X and $\emptyset$ only is also a topology on X , it is called the *indiscrete topology* or the *trivial topology*.

Let X be a set. Let J_f be a collection of all subsets U of X such that $X - U$ either is finite or is all of X . Then J_f is a topology on X , called the *finite complement* *topology*.

Result 0.1.2. J_f *is a finite complement topology.*

Proof. Since $X - X = \emptyset$ and $X - \emptyset = X$, either

is finite or is all of X. Both X and $\emptyset$ are in J_f.

To show that $\cup U_\alpha$ is in J_f.

$X - \cup U_\alpha = \cap (X - U_\alpha)$.

Since $X - U_\alpha$ is finite then $\cap (X - U_\alpha)$ is finite.

Then $(X - \cup U_\alpha)$ is finite.

Therefore, $\cup U_\alpha$ is in J_f.

If $U_1, U_2, \_\_\_, U_n$ or non empty elements of J_f.

To show that $\cap U_i$ is in J_f.

Now we know that $X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$.

since $(X - U_i)$ is finite then $\bigcup_{i=1}^{n} (X - U_i)$ is finite.

Then $\bigcap U_\alpha$ is in J_f.

Therefore, J_f is a finite complement topology.

Definition 0.1.3. Suppose that J and J' are two topologies on a given set X. If $J' \supset J$, we say that J' is *finer* than J; if J' properly contains J, we say that J' is *strictly finer* than J. We also say that J is *coarser* than J', or *strictly coarser*, in these two respective situations. We say J is *comparable* with J' if either $J' \supset J$ or $J \supset J'$.

4.2 BASIS FOR A TOPOLOGY

Definition 0.2.1. If X is a set, a *basis* for a topology on X is a collection B of subsets of X (called *basis elements*) such that

(i) For each $x \in X$, there is at least one basis element B containing x.

(ii) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the *topology* $\mathcal{J}$ *generated by* $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{J}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{J}$.

Lemma 0.2.2. *Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{J}$ on X. Then $\mathcal{J}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Let X be a set and $\mathcal{B}$ be the basis for the topology $\mathcal{J}$ on X. The collection of elements of $\mathcal{B}$ are also elements of $\mathcal{J}$ because $\mathcal{J}$ is a topology, their union is in $\mathcal{J}$.

Conversely, given $U \in \mathcal{J}$, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that

$x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$.

Lemma 0.2.3. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

Proof. First we prove that $\mathcal{C}$ is a basis.

Given $x \in X$, since X is an open set, by hypothesis an element C of $\mathcal{C}$ such that $x \in C \subset X$.

Let $x \in C_1 \cap C_2$ where C_1 and C_2 are the elements of $\mathcal{C}$. Since C_1

and C_2 are open, $C_1 \cap C_2$ are

open.

By hyphothesis, there exists an element C_3 of C such

that $x \in C_3 \subset C_1 \cap C_2$. Therefore, C is a basis.

Let J be the topology on X .

Let J' denote the

topology

generated by C .

To prove that $J' =$

J .

By 0.2.4, J' is finer than J .

Conversely, since each element of C is an element of J ,

the union of elements of C is also in J .

By 0.2.2, J' contains J .

Therefore, $J' = J$.

Therefore, C is a basis for the topology of X .

Lemma 0.2.4. *Let B and B' be bases for the topologies J and J' , respectively, on X . Then the following are equivalent:*

(i) J' is finer than J .

(ii) For each $x \in X$ and each basis element $B \in B$ containing x, there is a basis element $B' \in B'$ such that $x \in B' \subset B$.

Proof. To prove (ii)⇒(i)

Given an element $U \in J$.

To show that $U \in J'$.

Let $x \in U$. Since B generates J , there is an element $B \in B$ such that $x \in B \subset U$. By (ii), there exists an element $B' \in B'$ such that $x \in B' \subset B$, then $x \in B' \subset U$. By definition of basis for the topology, $U \in J'$.

To prove (i)⇒(ii)

Given $x \in X$ and $B \in \mathcal{B}$ with $x \in B$.

Now $B \in J$, by definition and $J \subset J'$ by (i); therefore $B \in J'$.

Since J' is generated by $\mathcal{B}'$, there is an element $B' \in \mathcal{B}'$

such that $x \in B' \subset B$.

Definition 0.2.5. If $\mathcal{B}$ is the collection of all open intervals in the real line,

$$(a, b) = \{x \mid a < x < b\},$$

the topology generated by $\mathcal{B}$ is called the *standard topology* on the real line.

If $\mathcal{B}'$ is the collection of all half-open intervals of the form

$$[a, b) = \{x \mid a \leq x < b\},$$

where $a < b$, the topology generated by $\mathcal{B}'$ is called the *lower limit topology* on R. When R is given the lower limit topology, we denote it by R_l . Finally let K denote the set of all numbers of the form $1/n$, for $n \in Z_+$, and let $\mathcal{B}''$ be the collection of all open intervals (a, b), along with all sets of the form $(a, b) - K$. The topology generated by $\mathcal{B}''$ will be called the *K-topology* on R. When R is given this topology, we denote it by R_k .

Lemma 0.2.6. *The topologies of* R_l *and* R_k *are strictly finer than three standard topology on* R*, but are not comparable with one another.*

Proof. Let J , J' , J'' be the topologies of R, R_l , R_k ,respectively. Given a basis element (a, b) for J and a point x of (a, b), the basis element $[x, b)$ for J' contains x and lies in $(a,$

b). On the otherhand,given the basis element [x, d) for $\mathcal{J}'$, there is no open interval (a, b) that contains x and lies in [x, d). Thus $\mathcal{J}'$ is strictly finer than $\mathcal{J}$.

Given a basis element (a, b) for $\mathcal{J}$ and a point x of (a, b), this same interval is a basis element for $\mathcal{J}''$ that contains x. On the otherhand, given the basis element $B = (-1, 1) - K$ for $\mathcal{J}''$ and the point 0 of B, there is no open interval that contains 0 and lies in B.

By definition of comparable, $\mathcal{J}'$ and $\mathcal{J}''$ are not comparable with one another.

Definition 0.2.7. *A subbasis* $\mathcal{S}$ *for a topology on* X *is a collection of subsets of* X *whose union equals* X. *The topology generated by the subbasis* $\mathcal{S}$ *is defined to be the collection* $\mathcal{J}$ *of all unions of finite intersections of elements of* $\mathcal{S}$.

4.3 THE ORDER TOPOLOGY

Definition 0.3.1. If X is a simply ordered set, there is a standard topology for X, defined using the order relation. It is called the *order topology*.

Suppose that X is a set having a simple order relation <. Given elements a and b of X such that a < b, there are four subsets of X that are called the *intervals* determined by a and b. They are the following:

$(a, b) = \{x \mid a < x < b\}$,

$(a, b] = \{x \mid a < x \leq b\}$, $[a, b) = \{x \mid a \leq x < b\}$, $[a, b] = \{x \mid a \leq x \leq b\}$.

A set of the first type is called an *open interval* in X

, a set of the last type is called a *closed interval* in X , and sets of the second and third types are called *half-open intervals*.

Definition 0.3.2. Let X be a set with a simple order relation; assume X has more than one element. Let B be the collection of all sets of the following types:

(1) All open intervals (a, b) in X .

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element(if any) of X .

(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element(if any) of X . The collection B is a basis for a topology on X , which is called the *order topology*.

Definition 0.3.3. If X is an ordered set, and a is an element of X , there are four subsets of X that are called *rays* determined by a. They are the following:

$(a, +\infty) = \{x | x > a\}$,

$(-\infty,a) = \{x | x < a\}$, $[a, +\infty) = \{x | x \geq a\}$, $(-\infty,a] = \{x | x \leq a\}$.

Sets of the first types are called *open rays*, and sets of the last two types are called *closed rays*.

Check In Progress-I

Q. 1 Define Order Topology.

Solution:

…………………………………………………………………………

…..………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

……………………

Q. 2 Basis of a Topology.

Solution:

……………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

4.4 THE PRODUCT TOPOLOGY ON $X \times Y$

Definition 0.4.1. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Theorem 0.4.2. *If* $\mathcal{B}$ *is a basis for the topology of* X *and* $\mathcal{C}$ *is a basis for the topology of* Y*, then the collection*

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Proof. We apply 0.2.3. Given an open set W of $X \times Y$ and a point $x \times y$ of W, by definition of the product topology there is a basis element $U \times V$ such that

$x \times y \in U \times V \subset W$.

Because $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y respectively, we can choose an element B of $\mathcal{B}$ such that $x \in B \subset U$ and an element C of $\mathcal{C}$ such that $y \in C \subset V$. Then $x \times y \in B \times C \subset W$.

Therefore, D is a basis for $X \times Y$.

Definition 0.4.3. Let $\pi_1 : X \times Y \to X$ be defined by the equation

$\pi_1(x, y) = x$;

let $\pi_2 : X \times Y \to Y$ be defined by the equation

$\pi_2(x, y) = y$.

The maps π_1 and π_2 are called the *projections* of $X \times Y$ onto its first and second factors, respectively.

We use the word ”onto” because π_1 and π_2 are surjective.

Note If U is an open subset of X , then the set $\pi^{-1}_1 (U)$ is precisely the set

$U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y , then

$\pi^{-1}_2 (V) = X \times V$,

which is also open in $X \times Y$. The intersection of these two sets is the

set $U \times V$. Theorem 0.4.4. *The collection*

$$S = \{\pi^{-1}_1(U) \mid U \text{ open in } X\} \cup \{\pi^{-1}_2(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let J denote the product topology on $X \times Y$.

Let J' be the topology generated by S . Because every element of S belongs to

J , by definition of subbasis, arbitrary unions of finite intersections of elements of S . Thus $J' \subset J$. On the otherhand,

$$U \times V = \pi^{-1}_1 (U) \cap \pi^{-1}_2 (V)$$

where $\pi^{-1}_1(U)$ is open in X and $\pi^{-1}_2 (V)$ is open in Y .

Since $U \times V \in J$, we have $U \times V = \pi^{-1}_1 (U) \cap \pi^{-1}_2 (V)$. $U \times V \in J'$. Therefore,

$\mathcal{J} \subset \mathcal{J}'$.

4.5 THE SUBSPACE TOPOLOGY

Definition 0.5.1. Let X be a topological space with topology $\mathcal{J}$. If Y is a subset of X, the collection

$$\mathcal{J}_Y = \{Y \cap U \mid U \in \mathcal{J}\}$$

is a topology on Y, called the *subspace topology*. With this topology, Y is called a *subspace* of X; its open sets consist of all intersections of open sets of X with Y.

Lemma 0.5.2. *If* $\mathcal{B}$ *is a basis for the topology of* X *then the collection*

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Consider U is open in X. Given $\mathcal{B}$ is a basis for the topology of X. We can choose an element B of $\mathcal{B}$ such that $y \in B \subset U$.

Then $y \in B \cap Y \subset U \cap Y$, since $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$.

By 0.2.3 or definition of basis, $\mathcal{B}_Y$ is a basis for the subspace topology on Y.

Definition 0.5.3. If Y is a subspace of X, we say that a set U is *open in* Y (or open *relative* to Y) if it belongs to the topology of Y; this implies in particular that it is a subset of Y. We say that U is *open in* X if it belongs to the topology of X.

Lemma 0.5.4. *Let* Y *be a subspace of* X. *If* U *is open in* Y *and* Y *is* *open in* X*, then* U *is open in* X .

Proof. Given U is open in Y and Y is open in X .

Since U is open in Y and Y is a subspace of X then $U = Y \cap V$ where V is open in X .

Since Y and V are both open in X , $Y \cap V$ is open in X .

Therefore, U is open in X .

Theorem 0.5.5. *If* A *is a subspace of* X *and* B *is a subspace of* Y *, then the product topology on* $A \times B$ *is the same as the topology* $A \times B$ *inherits as a subspace of* $X \times Y$.

Proof. The set $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y .

Then $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$. Now

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B).$$

Since $U \cap A$ and $V \cap B$ are the general open sets for the subspace topologies on A and B respectively, the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product on $A \times B$.

The bases for the subspace topology on $A \times B$ and for the product topology on

$A \times B$ are the same. Hence the topologies are the same.

Theorem 0.5.6. *Let* X *be an ordered set in the order topology; let* Y *be a subset of* X *that is convex in* X *. Then the order topology on* Y *is the same as the topology* Y *inherits as a subspace of* X .

Proof. Consider the ray $(a, +\infty)$ in X .

If $a \in Y$, then $(a, +\infty) \cap Y = \{x | x \in Y \text{ and } x > a\}$; this is an open ray of the ordered set Y .

If $a \notin Y$, then a is either a lower bound on Y or an upper bound on Y , since Y is convex.

If $a \in Y$, the set $(a, +\infty) \cap Y$ equals all of Y. If $a \notin Y$, it is empty.

Similarly the intersection of the ray $(-\infty, a) \cap Y$ is either an open ray of Y, or Y itself or empty.

Since the sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a sub basis for the subspace topology on Y and since each is open in the order topology, the order topology.

contains the subspace topology.

Conversely, Y equals the intersection of X with Y, that is $X \cap Y = Y$. So it is open in the subspace topology on Y. The order topology is contained in the subspace topology. Therefore, the order topology and subspace topology are

same.

4.6 CLOSED SETS AND LIMIT POINTS

Definition 0.6.1. A subset A of a topological space X is said to be *closed* if the set $X - A$ is open.

Theorem 0.6.2. *Let X be a topological space. Then the following conditions hold:*

(1) $\emptyset$ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

Proof. (1) $\emptyset$ and X are closed because they are the complements of the open set X and $\emptyset$ respectively.

(2) Consider a collection of closed sets $\{A_\alpha\}_{\alpha\in J}$, we apply De Morgan's law,

$$X - \bigcap_{\alpha\in J} A_\alpha = \bigcup_{\alpha\in J} (X - A_\alpha)$$

Since the sets $X - A_\alpha$ are open. By definition of closed sets, the right side of this equation represents an arbitrary union of open sets and is thus open. Therefore, $\bigcap A_\alpha$ is closed.

(3) Similarly, if A_i is closed for $i = 1, 2, \_\_\_, n$. Consider the equation

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i)$$

The set on the right side of this equation is a finte intersection of open sets and

is therefore open.

Hence $\bigcup_{i=1}^{n} A_i$ is closed.

Definition 0.6.3. If Y is a subspace of X, we say that a set A is *closed in* Y if A is a subset of Y and if A is closed in the subspace topology of Y (that is, if $Y - A$ is open in Y).

Theorem 0.6.4. *Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.*

Proof. Assume that $A = C \cap Y$, where C is closed in X. Then $X - C$ is open in X, so that $(X - C) \cap Y$ is open in Y. By the definition of the subspace topology, but $(X - C) \cap Y = Y - A$. Hence $Y - A$ is open in Y, so that A is closed in Y. Conversely, assume that A is closed in Y. Then $Y - A$ is open in Y. By definition, it equals the intersection of an open set U of X with Y. The set $X - U$ is closed in X and $A = Y \cap (X - U)$. Hence A equals the intersection of a closed set of X with Y.

Theorem 0.6.5. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. Given A is closed in Y and Y is closed in X. Since A is closed in Y and Y is a subspace of X,

let $A = Y \cap (X - B)$ where $X - B$ is open in X. Then B is closed in X.

Since Y and B are both closed in X . Then $Y \cap (X - B)$ is closed in X . Therefore, A is closed in X .

Definition 0.6.6. Given a subset A of a topological space X , the *interior* of A is defined as the union of all open sets contained in A, and the *closure* of A is defined as the intersection of all closed sets containing A.

The interior of A is denoted by Int A and the closure of A is denoted by Cl A or by $\bar{A}$. Obviously Int A is an open set and $\bar{A}$ is a closed set; furthermore,

Int $A \subset A \subset \bar{A}$.

If A is open, A=Int A; while if A is closed, $\bar{A} = A$.

Theorem 0.6.7. *Let Y be a subspace of X ; let A be a subset of Y ; let $\bar{A}$ denote the closure of A in X . Then the closure of A in Y equals $\bar{A} \cap Y$.*

Proof. Let B denote the closure of A in Y . The set $\bar{A}$ is closed in X , so $\bar{A} \cap Y$ is closed in Y . By 0.6.4, since $\bar{A} \cap Y$ contains A and since B is closed. By definition B equals the intersection of all closed subsets of Y containing A, we must have $B \subset (\bar{A} \cap Y)$.

On the otherhand, we know that B is closed in Y . By 0.6.4, $B = C \cap Y$ for some set C closed in X . Then C is a closed set of X containing A; because $\bar{A}$ is the intersection of all such closed sets, we conclude that $\bar{A} \subset C$. Then $(\bar{A} \cap Y) \subset (C \cap Y) = B$. Therefore, $B = \bar{A} \cap Y$.

Theorem 0.6.8. *Let A be a subset of the topological space X .*

(a) Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A.

Notes

Proof. (a)We prove this theorem by contrapositive method.

If x is not in A, since A is closed, $\bar{A} = A$. The set $U = X - \bar{A}$ is an open set containing x that does not intersect A.

Conversely, if there exists an open set U containing x which does not intersect

A. Then $X - U$ is a closed set containing A.

By definition of the closure of A, the set $X - U$ must contain A, since $x \in U$. Therefore, x cannot be in A.

(b) Write the definition of topology generated by basis, if every open set in X intersects A, so does every basis element B containing *x*, because B is an open set.

Conversely, if every basis element containing x intersects A, so does every open set U containing x, because U contains a basis element that contains x.

Definition 0.6.9. If A is a subset of the topological space X and if x is a point of X , we say that x is a *limit point* (or "cluster point" or "point of accumulation") of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$. The point x may lie in A or not; for this definition it does not matter.

Theorem 0.6.10. *Let* A *be a subset of the topological space* X *; let* A' *be the set of all limit points of* A*. Then* $\bar{A} = A \cup A'$.

Proof. Let A' be the set of all limit points of A.

If $x \in A'$, every neighborhood of x intersects A in a point different from x. By 0.6.8, $x \in A$. Then $A' \subset A$.

By definition of closure, $A \subset \bar{A}$. Therefore, $A \cup A' \subset \bar{A}$.

Conversely, let $x \in A$

To show that $\bar{A} \subset A \cup A'$

If $x \in A$ then it is trivially true for $x \in A \cup A'$.

Suppose $x \notin A$. Since $x \in \bar{A}$, by 0.6.8, we know that every neighborhood U of x intersects A, because $x \notin A$, the set U must intersect A in a point different from

x. Then $x \in A'$ so that $x \in A \cup A'$.

Then $\bar{A} \subset A \cup A'$.

Therefore, $\bar{A} = A \cup A'$.

Corollary 0.6.11. *A subset of a topological space is closed if and only if it contains all its limit points.*

Proof. The set A is closed if $\bar{A} = A$. By 0.6.10, $A' \subset A$. Therefore A contqain all its limit P points. Conversely, Suppose A′ C A.

Then A∪A′CA. Then true $\bar{A}$ C A. But $A \subseteq \bar{A}$: $\bar{A} \supset A$. Therefore A is α

Definition 0.6.12. A topological space X is called a *Hausdorff space* if for each pair x_1, x_2 of distinct points of X , there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively, that are disjoint.

Theorem 0.6.13. *Every finite point set in a Hausdorff space* X *is closed.*

Proof. It is enough to show that every one-point set $\{x_0\}$ is closed.

If x is a point of X different from x_0, then x and x_0 have disjoint neighborhoods U and V respectively.

Since U does not intersect $\{x_0\}$, the point x cannot belong to the closure of the set $\{x_0\}$.

As a result, the closure of the set $\{x_0\}$ is $\{x_0\}$ itself.

Therefore, $\{x_0\}$ is closed.

Note: The condition that finite point sets be closed is in fact weaker than the Hausdro condition. For example, the real line R in the finite complement topology is not a Hausdor space, but it is a space in which finite point sets are closed. The condition that finite point sets be closed has been given a name of its own; it is called the T_1 *axiom*.

Theorem 0.6.14. *Let* X *be a space satisfying the* T_1 *axiom; let* A *be a subset of*

X. *Then the point* x *is a limit point of* A *if and only if every neighborhood of* x *contains infinitely many points of* A.

Proof. If every neighborhood of x intersects A in infinitely many points, it certainly intersects A in some point other than x itself, so that x is a limit point of

A.

Conversely, suppose that x is a limit point of A and suppose some neighborhood U of x intersects A in only finitely many points.

Let $\{x_1, x_2, \_\_\_, x_m\}$ be the points of $U \cap (A - \{x\})$.

The set $X - \{x_1, x_2, \_\_\_, x_m\}$ is an open set of X, since the finite point set $\{x_1, x_2, \_\_\_, x_m\}$ is closed then

$U \cap (X - \{x_1, x_2, \_\_\_, x_m\})$

is a neighborhood of x that does not intersect the set $A - \{x\}$. Since $\{x_1, x_2, \_\_\_, x_m\}$ be points of $U \cap (A - \{x\})$.

This contradicts the assumption that x is a limit point of A.

Theorem 0.6.15. *If* X *is a Hausdorff space, then a sequence of points of* X *converges to at most one point of* X .

Proof. Suppose that x_n is a sequence of points of X that converges to x.

If $y \neq x$, let U and V be disjoint neighborhoods of x and y respectively.

Since U contains x_n for all but finitely many values of n, the set V cannot contains x_n. Therefore, x_n cannot converge.

If the sequence x_n of points of the Hausdorff space X converges to the point x of X, we often write $x_n \to x$.

Therefore, x is the limit of the sequence x_n.

Theorem 0.6.16. *Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.*

Proof. Let X and Y be two Hausdorff spaces. To prove $X \times Y$ is Hausdorff .

Let $x_1 \times y_1$ and $x_2 \times y_2$ be two distinct points of $X \times Y$. Then x_1, x_2 are distinct points of X and X is a Hausdorff space, there exists neighborhood U_1 and U_2 of x_1 and x_2 such that $U_1 \cap U_2 = \emptyset$

Similarly, y_1 , y_2 are distinct point of Y and Y is a Hausdor space, there exists neighborhood V_1 and V_2 of y_1 and y_2 such that $V_1 \cap V_2 = \emptyset$.

Then clearly $U_1 \times V_1$ and $U_2 \times V_2$ are open sets in $X \times Y$ containing $x_1 \times y_1$ and

$x_2 \times y_2$ such that $(U_1 \times V_1) \cap (U_2 \times V_2) = \emptyset$.

Therefore, $X \times Y$ is a Hausdorff space.

Let X be a Hausdorff space and let Y be a subspace. To prove Y is a Hausdorff space.

Let y_1, y_2 be two distinct points of Y . Then y_1 and y_2 are distinct points in X and X is Hausdorff there exists neighborhood U_1 and U_2 of y_1 and y_2 such that $U_1 \cap U_2 = \emptyset$. Then $U_1 \cap Y$ and $U_2 \cap Y$ are distinct neighborhood of y_1 and y_2 in Y .

Therefore, Y is a Hausdorff space.

Check In Progress-II

Q. 1 Define Closed Set .

Solution :

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

Q. 2 Define Subspace Topology.

Solution :

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

4.7 SUBBASE

In topology, a subbase (or subbasis) for a topological space *X* with topology *T* is a subcollection *B* of *T* that generates *T*, in the sense that *T* is the smallest topology containing *B*. A slightly different definition is used by some authors, and there are other useful equivalent formulations of the definition; these are discussed below.

Let *X* be a topological space with topology *T*. A subbase of *T* is usually defined as a subcollection *B* of *T* satisfying one of the two following equivalent conditions:

1. The subcollection *B* *generates* the topology *T*. This means that *T* is the smallest topology containing *B*: any topology *T′* on *X* containing *B* must also contain *T*.
2. The collection of open sets consisting of all finite intersections of elements of *B*, together with the set *X*, forms a basis for *T*. This means that every proper open set in *T* can be written as a union of finite intersections of elements of *B*. Explicitly, given a point *x* in an open set $U \subseteq X$, there are finitely many sets $S_1, \ldots, S_n$ of *B*, such that the intersection of these sets contains *x* and is contained in *U*.

(Note that if we use the nullary intersection convention, then there is no need to include *X* in the second definition.)

For *any* subcollection *S* of the power set P(*X*), there is a unique topology having *S* as a subbase. In particular, the intersection of all topologies on *X* containing *S* satisfies this condition. In general, however, there is no unique subbasis for a given topology.

Thus, we can start with a fixed topology and find subbases for that topology, and we can also start with an arbitrary subcollection of the power set P(X) and form the topology generated by that subcollection. We can freely use either equivalent definition above; indeed, in many cases, one of the two conditions is more useful than the other.

Alternative Definition

Sometimes, a slightly different definition of subbase is given which requires that the subbase B cover X.[1] In this case, X is the union of all sets contained in B. This means that there can be no confusion regarding the use of nullary intersections in the definition.

However, with this definition, the two definitions above are not always equivalent. In other words, there exist spaces X with topology T, such that there exists a subcollection B of T such that T is the smallest topology containing B, yet B does not cover X. In practice, this is a rare occurrence; e.g. a subbase of a space that has at least two points and satisfies the T_1 separation axiom must be a cover of that space.

Examples

The usual topology on the real numbers R has a subbase consisting of all semi-infinite open intervals either of the form $(-\infty,a)$ or (b,∞), where a and b are real numbers. Together, these generate the usual topology, since the intersections $(a,b) = (-\infty,b) \cap (a,\infty)$ for $a < b$ generate the usual topology. A second subbase is formed by taking the subfamily where a and b are rational. The second subbase generates the usual topology as well, since the open intervals (a,b) with a, b rational, are a basis for the usual Euclidean topology.

The subbase consisting of all semi-infinite open intervals of the form $(-\infty,a)$ alone, where a is a real number, does not generate the usual topology. The resulting topology does not satisfy the T_1 separation axiom, since all open sets have a non-empty intersection.

The initial topology on X defined by a family of functions $f_i : X \to Y_i$, where each Y_i has a topology, is the coarsest topology on X such that each f_i is continuous. Because continuity can be defined in terms of the inverse images of open sets, this means that the initial topology on X is given by taking all $f_i^{-1}(U)$, where U ranges over all open subsets of Y_i, as a subbasis.

Two important special cases of the initial topology are the product topology, where the family of functions is the set of projections from the product to each factor, and the subspace topology, where the family consists of just one function, the inclusion map.

The compact-open topology on the space of continuous functions from X to Y has for a subbase the set of functions

where $K \subseteq X$ is compact and U is an open subset of Y.

Results using subbase

One nice fact about subbases is that continuity of a function need only be checked on a subbase of the range. That is, if B is a subbase for Y, a function $f : X \to Y$ is continuous iff $f^{-1}(U)$ is open in X for each U in B.

4.8 ALEXANDER SUBBASE THEOREM

There is one significant result concerning subbases, due to James Waddell Alexander II.

Alexander Subbase Theorem. Let X be a topological space with a subbasis B. If every cover by elements from B has a finite subcover, then the space is compact.

Note that the corresponding result for basic covers is trivial.

Proof Outline: Assume by way of contradiction that the space X is not compact, yet every subbasic cover from B has a finite subcover.
Use Zorn's Lemma to find an open cover C without finite subcover that is *maximal* amongst such covers. That means that if V is an open set of X which is not in C, then $C \cup \{V\}$ has a finite subcover, necessarily of the form $\{V\} \cup C_V$, where the choice of the finite subset C_V of the cover C depends on the picked additional set V .
Consider $C \cap B$, that is, the subbasic subfamily of C. We claim $C \cap B$ does not cover X. If it covered X, then it would be a cover from elements of B and by hypothesis on B, it would have a finite

subcover from $C \cap B$ which is at the same time also a finite subcover from C. But from definition of C ,C does not have a finite subcover of X, so $C \cap B$ does not cover X. So there exists an element x from X but uncovered by $C \cap B$. C covers X (with infinite number of open sets), so $x \in U$ for some $U \in C$. B is a subbasis, so for some $S_1, ..., S_n \in B$, we have: $x \in S_1 \cap \cdots \cap S_n \subseteq U$.

Since x is uncovered by $C \cap B$, $S_i \notin C$ for each i. (If $S_i \in C$ for some i, then it would hold $S_i \in C \cap B$ and since $x \in S_i$, $C \cap B$ would also cover point x, contrary to its choice). As noted above from the *maximality* of the cover C, for each i there exists a finite subset C_{Si} of cover C such that $\{S_i\} \cup C_{Si}$ forms a finite cover of X. Let's denote C_F the finite union of the finite sets C_{Si} where i iterates from 1 to n. Then for each i the former finite cover of X can be replaced by a new bigger and still finite cover $\{S_i\} \cup C_F$ of X. The finite set $\{S_i\} \cup C_F$ covers X for each i, so also $\{S_1 \cap \cdots \cap S_n\} \cup C_F$ covers X. The intersection in the cover can be replaced by the single *bigger* open set U from cover C. So $\{U\} \cup C_F$ is also a finite cover of X and made of the open sets only from C.
Thus C has a finite subcover of X, in contradiction to the choice of C.
Therefore the original assumption of X not being compact is wrong due to a contradiction we reached. Therefore X is compact. Q.E.D.

Although this proof makes use of Zorn's Lemma, the proof does not need the full strength of choice. Instead, it relies on the intermediate Ultrafilter principle.

Using this theorem with the subbase for R above, one can give a very easy proof that bounded closed intervals in R are compact.

Tychonoff's theorem, that the product of compact spaces is compact, also has a short proof. The product topology on $\prod_i X_i$ has, by definition, a subbase consisting of *cylinder* sets that are the inverse projections of an open set in one factor. Given a *subbasic* family C of the product that does not have a finite subcover, we can partition $C = \cup_i C_i$ into subfamilies that consist of exactly those cylinder sets corresponding to a given factor space. By assumption, no C_i has a finite subcover. Being cylinder sets, this means their projections onto X_i have no finite subcover, and since each X_i is compact, we can find a point $x_i \in X_i$ that is not covered by the projections of C_i onto X_i. But then $(x_i)_i \in \prod_i X_i$ is not covered by C.

Note, that in the last step we implicitly used the axiom of choice (which is actually equivalent to Zorn's lemma) to ensure the existence of $(x_i)_i$.

4.9. TOPOLOGICAL BASIS AND SUB-BASIS

Definition 1 Topological space X is a set with a specific collection T of subsets called open sets with the following properties.

1. ∅, X ∈ T .

2. A union of open sets is open.

3. A finite intersection of open sets is open.

Examples 1. A metric space with the usual open sets.

2. Let X be a set. Then T = P(X) is called the discrete topology and T = {∅, X} the indiscrete topology.

3. X = {a, b}. Then T = {∅, X, {a}} is a topology.

4. Let X be an infinite set. Then T = {U ⊂ X|U c is a f inite set} ∪ {∅} is called cofinite topology.

Definition 2 Let X and Y be topological spaces. Then f : X → Y is continuous if f −1 (V) is open for all V open in Y .

Definition 3 A collection B of open sets of a topological space X is called a basis if each open set in X can be represented as a union of elements of B.

Proposition 1 Suppose that a collection B of subsets of a set X satisfies the following two properties:

1. The elements of B cover X, i.e., X = ∪B∈BB.

2. If x belongs to two elements B1 and B2 of B, then there exists $B \in B$ such that $x \in B \subset B1 \cap B2$. Then the collection T (B) of all unions of elements of B defines a topology on a set X(called the topology generated by B). Proof is easy.

Examples 1. Let $B = \{(a, b)|a < b$, a and b real$\}$. Then (R, T (B)) is called the usual topology of R.

2. Let $B = \{[a, b)|a < b$, a and b real$\}$. Then Rl = (R, T (B)) is called the real line with half-open topology. 1 3. Let $B = \{[a, b]|a \leq b$, a and b real$\}$. Then (R, T (B)) becomes a discrete space since $[a, a] = \{a\}$.

Proposition 2 Let B and B 0 be basis for the topology T and T 0 , respectively on X. Then $T \subset T$ 0 if and only if for each $B \in B$ and $x \in B$, there is a basis element B $0 \in B0$ such that $x \in B$ $0 \subset B$. In this case, T 0 is said to be finer than T . Proof ($\Rightarrow$) For each B in B, $B \in B \subset T$ (B) $\subset T$ (B 0). Therefore $B \in T$ (B 0). Since T (B 0) is generated by B 0 , for each $x \in B$ there is an element B $0 \in B0$ such that $x \in B$ $0 \subset B$. ($\Leftarrow$) Let U be an element of T and $x \in U$. Since B generates T , there is an element $B \in B$ such that $x \in B \subset U$. Then there is B 0 such that $x \in B$ 0 $\subset B \subset U$ and hence $U \in T$ 0 .

Examples 1. Let $B = \{(a, b)|a < b$, a and b real$\}$ and B 0 = $\{[a, b)|a < b$, a and b real$\}$. Then T (B) T (B 0)

3. Let B be the collection of all open discs in the plane and B 0 is the collection of open squares. Then T = T 0 Homework
4. 3 Let C([0, 1]) be the collection of continuous functions on [0, 1]. Consider the following topologies. T1 = the topology induced by L1 norm, i.e., kfk1 = R 1 0 |f| T2 = the topology induced by L2 norm, i.e., kfk2 = (R 1 0 |f| 2) 1/2 T∞ = the topology induced by L∞ norm, i.e., kfk∞ = sup|f| stopology[þt_ ☐©⁇ ñ⁇í⁇\¦ [O"î .

Definition 4 Let X be a topological space. A collection S of open sets is called a subbasis if each open set in X can be written as a union of finite intersections of elements of S.

Proposition 3 Let S be a collection of subsets of a set X whose union is X. Then the collection of all unions of finite intersections of sets in S form a topology for X. This topology T (S) will be called the topology generated by S, and T (S) is the smallest topology containing S. (Proof is easy using Prop 1.)

Examples 1. $S = \{(a,\infty),(-\infty, b)\}$ is a subbasis for the standard topology of R. 2

2. $S = \{[a,\infty),(-\infty, b)\}$ is a subbasis for Rl .

3. $S = \{R - \{p\}|p \in R\}$ is a subbasis for the cofinite topology.

4.10 SUMMARY

We study in this unit about subspace topology and its lemma. We study in this unit closed set and limit point of subspace topology. We study topological space of subspace topology.

4.11 KEYWORD

SUBSPACE : A space that is wholly contained in another space, or whose points or elements are all in another space

SPACE : A continuous area or expanse which is free, available, or unoccupied

TYCHONOFF'S : A topological space is termed a **Tychonoff** space if it satisfies the following equivalent conditions

4.12 EXERCISE

Q. 1 A metric space with the usual open sets.

Q. 2 A subset of a topological space is closed if and only if it contains all its limit points

Q. 3 The topologies of R_l and R_k are strictly finer than three standard topology on R, but are not comparable with one another.

Q. 4 Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X . Then the order topology on Y is the same as the topology Y inherits as a subspace of X

Q. 5 If A is a subspace of X and B is a subspace of Y , then the product topology on A ×B is the same as the topology A ×B inherits as a subspace of X × Y .

4.13 ANSWER FOR CHECK IN PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 2

Q 2 Check in Section 3

Check in Progress-II

Answer Q. 1 Check in Section 6

Q 2 Check in Section 5

4.14 SUGGESTION READING AND REFERENCES

1. *Merrifield, Richard E.; Simmons, Howard E. (1989). Topological Methods in Chemistry. John Wiley & Sons. p. 17. ISBN 0-471-83817-9. Retrieved 13 June 2013. A collection S of subsets that satisfies criterion (i) is called a subbasisfor a topology on X.*

Notes

- *Willard, Stephen (2004), General topology, New York: Dover Publications, ISBN 978-0-486-43479-7, MR 2048350*
- *Armstrong, M. A. (1983) [1979]. Basic Topology. Undergraduate Texts in Mathematics. Springer. ISBN 0-387-90839-0.*
- Bredon, Glen E., *Topology and Geometry* (Graduate Texts in Mathematics), Springer; 1st edition (October 17, 1997). ISBN 0-387-97926-3.
- Bourbaki, Nicolas; *Elements of Mathematics: General Topology*, Addison-Wesley (1966).
- Brown, Ronald, *Topology and Groupoids*, Booksurge (2006) ISBN 1-4196-2722-8 (3rd edition of differently titled books)
- Čech, Eduard; *Point Sets*, Academic Press (1969).
- Fulton, William, *Algebraic Topology*, (Graduate Texts in Mathematics), Springer; 1st edition (September 5, 1997). ISBN 0-387-94327-7.
- *Gallier, Jean; Xu, Dianna (2013). A Guide to the Classification Theorem for Compact Surfaces. Springer.*
- Gauss, Carl Friedrich; *General investigations of curved surfaces*, 1827.
- Lipschutz, Seymour; *Schaum's Outline of General Topology*, McGraw-Hill; 1st edition (June 1, 1968). ISBN 0-07-037988-2.
- Munkres, James; *Topology*, Prentice Hall; 2nd edition (December 28, 1999). ISBN 0-13-181629-2.
- Runde, Volker; *A Taste of Topology (Universitext)*, Springer; 1st edition (July 6, 2005). ISBN 0-387-25790-X.
- *Schubert, Horst (1968), Topology, Macdonald Technical & Scientific, ISBN 0-356-02077-0*
- Steen, Lynn A. and Seebach, J. Arthur Jr.; *Counterexamples in Topology*, Holt, Rinehart and Winston (1970). ISBN 0-03-079485-4.
- *Vaidyanathaswamy, R. (1960). Set Topology. Chelsea Publishing Co. ISBN 0486404560.*

4.15 BIBLIOGRAPHY

- Hazewinkel, Michiel, ed. (2001) [1994], "Topological space", Encyclopedia of Mathematics, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4
- "Topological space". PlanetMath.

UNIT 5 INTERIOR AND BOUNDARY POINTS OF A SET

STRUCTURE

5.0 OBJECTIVE

- We learn in this unit boundary and exterior point in Euclidean Space
- Learn open and closed set
- Learn Exterior and Interior Operator
- Learn Subspace terminology
- Learn Identification Topology

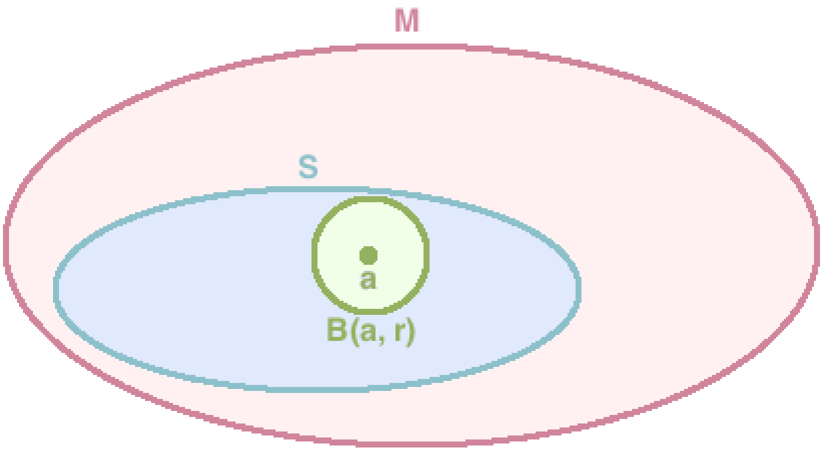
Learn boundary Topology

5.1. INTRODUCTION

Recall from the Interior, Boundary, and Exterior Points in Euclidean Space that if $S \subseteq \mathbb{R}n$ then a point $\mathbf{a} \in S$ is called an interior point of S if there exists a positive real number $r>0$ such that the ball centered at a with radius r is a subset of S.
Furthermore, a point $\mathbf{a}$ is called a boundary point of S if for every positive real number $r>0$ we have that there exists points $\mathbf{x},\mathbf{y} \in B(\mathbf{a},r)$ such that $\mathbf{x} \in S$ and $\mathbf{y} \in Sc$.
We will now generalize these definitions to metric spaces (M,d).

Definition: Let (M,d) be a metric space and let $S \subseteq M$. A point $a \in S$ is said to be an **Interior Point** of S if there exists a positive real number $r>0$such
that the ball centered at a with radius r with respect to the metric d is a subset
of S, i.e., $B(a,r) \subseteq S$. The set of all interior points of S is called the **Interior** of S
and is denoted $\mathrm{int}(S)$

In shorter terms, a point $a \in S$ is an interior point of S if there exists a ball centered at a that is fully contained in S. Note that from the definition above we have that a point can be an interior point of a set only if that point is contained in S. Therefore $\mathrm{int}(A) \subseteq A$.



The point a is an interior point of S.

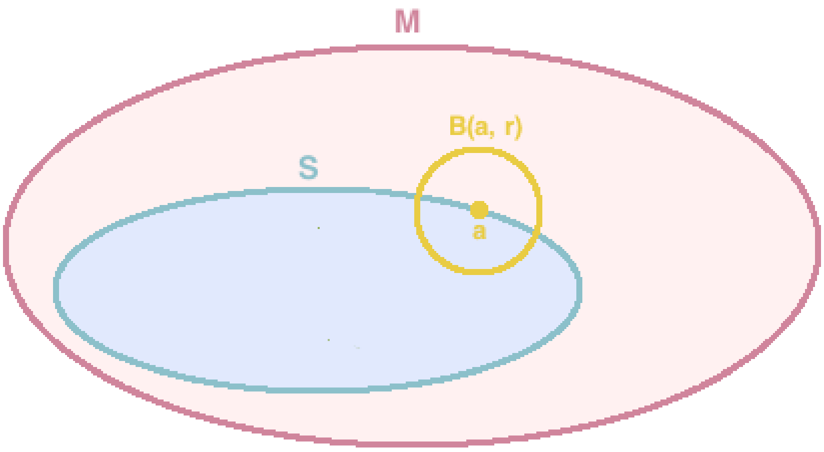
Definition: Let (M,d) be a metric space and let $S\subseteq M$. A point $a\in M$ is said to be a

Boundary Point of S if for every positive real number $r>0$we have that there exists

points $x,y\in B(a,r)$ such that $x\in S$ and $y\in Sc$. The set of all boundary points is called

the **Boundary** of S and is denoted ∂S or bdry(S).

A point $a\in M$ is said to be a boundary point of S if every ball centered at a contains points in S and points in the complement Sc. Notice that from the definition above that a boundary point of a set need not be contained in that set.



The point a is a boundary point of S.

5.2 INTERIOR POINTS, OPEN AND CLOSED SETS

Let (X,d) be a metric space with distance $d:X\times X\to[0,\infty)$.

- A point $x_0 \in D \subset X$ is called an **interior point in D** if there is a small ball centered at x_0 that lies entirely in D,

$$x_0 \text{ interior point } \overset{\text{def}}{\Longleftrightarrow} \exists \varepsilon > 0;\, B_\varepsilon(x_0) \subset D.$$

- A point $x_0 \in X$ is called a **boundary point of D** if any small ball centered at x_0 has non-empty intersections with both D and its complement,

$$x_0 \text{ boundary point } \overset{\text{def}}{\Longleftrightarrow} \forall \varepsilon > 0\, \exists x, y \in B_\varepsilon(x_0);\, x \in D, y \in X \setminus D.$$

- The set of interior points in D constitutes its **interior**, int(D), and the set of boundary points its **boundary**, ∂D. D is said to be **open** if any point in D is an interior point and it is **closed** if its boundary ∂D is contained in D; the **closure of D** is the union of D and its boundary:

$$\overline{D} := D \cup \partial D.$$

Alternative notations for the closure of D in X include D$\overline{X}$
, clos(D) and clos(D;X).

Example

In R with the usual distance $d(x,y)=|x-y|d(x,y)=|x-y|$, the interval (0,1) is open, [0,1) neither open nor closed, and [0,1]closed.[2)]

- The set

$$D := \{(x,y) \in R^2 : x > 0, y \geq 0\}$$

is neither closed nor open in Euclidean space R2 (metric coming from a norm, e.g., $d(x,y)=\|x-y\|l2=[((x1-y1)+(x2-y2)2)]$ ½, since its boundary contains both points (x,0), x>0, in D and points (0,y), y≥0y≥0, not in DD. The closure of D is

$$\bar{D} = x,y)\in R2:x\geq 0,y\geq 0\}.$$

- An entire metric space is both open and closed (its boundary is empty).
- In l∞l∞,

$$B1\ni(1/2,2/3,3/4,\ldots)\in B.\ B1\not\ni(1/2,2/3,3/4,\ldots)\in \overline{B1}.$$

- For a general metric space, the **closed ball**

$$B\sim r(x0):=\{x\in X:d(x,x0)\leq r\}$$

may be larger than the closure of a ball, Br(x0) .If we let X be a space with the discrete metric,

$$d(x,x)=0,d(x,y)=1,x\neq y.$$

Then

B1(x0)={x0}, so that $\overline{B1(x0)}=\overline{\{x0\}}=\{x0\}$.B1(x0)={x0}, so that $\overline{B1(x0)}=\overline{\{x0\}}=\{x0\}$.

But

$$B\sim 1(x0)=X.B\sim 1(x0)=X.$$

℘ (Open) balls are open

Let (X,d)(X,d) be a metric space, x0x0 a point in XX, and r>0r>0. Then Br(x0)Br(x0) is open in XX with respect to the metric dd.

Check in Progress-I

Q 1. Define Interior Point.

Solution :

………………………………………………………………………………….

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

Q. 2 Define Exterior Point.

Solution .

………………………………………………………………………………

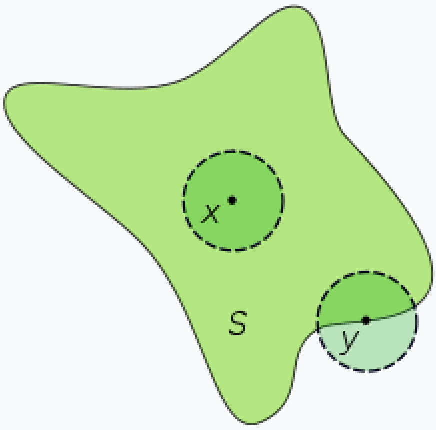
………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

5.3 INTERIOR (TOPOLOGY)



The point *x* is an interior point of *S*. The point *y* is on the boundary of *S*.

In mathematics, specifically in topology, the **interior** of a subset of a topological space is the union of all open subsets of that set. A point that is in the interior of *S* is an **interior point** of *S*.

The interior of *S* is the complement of the closure of the complement of *S*. In this sense interior and closure are dual notions.

The **exterior** of a set is the interior of its complement, equivalently the complement of its closure; it consists of the points that are in neither the set nor its boundary. The interior, boundary, and exterior of a subset together partition the whole space into three blocks (or fewer when one or more of these is empty). The interior and exterior are always openwhile the boundary is always closed. Sets with empty interior have been called **boundary sets**.[1]

5.3.1 Interior Point

If *S* is a subset of a Euclidean space, then *x* is an interior point of *S* if there exists an open ball centered at *x* which is completely contained in *S*. (This is illustrated in the introductory section to this article.)

This definition generalizes to any subset *S* of a metric space *X* with metric *d*: *x* is an interior point of *S* if there exists $r > 0$, such that *y* is in *S* whenever the distance $d(x, y) < r$.

This definition generalises to topological spaces by replacing "open ball" with "open set". Let *S* be a subset of a topological space *X*. Then *x* is an interior point of *S* if *x* is contained in an open subset of *X* which is completely contained in *S*. (Equivalently, *x* is an interior point of *S* if *S* is a neighbourhood of *x*.)

5.3.2 Interior of a Set

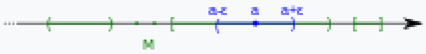
The **interior** of a set *S* is the set of all interior points of *S*. The interior of *S* is denoted int(*S*), Int(*S*) or S^o. The interior of a set has the following properties.

- int(*S*) is an open subset of *S*.
- int(*S*) is the union of all open sets contained in *S*.
- int(*S*) is the largest open set contained in *S*.
- A set *S* is open if and only if *S* = int(*S*).
- int(int(*S*)) = int(*S*) (idempotence).
- If *S* is a subset of *T*, then int(*S*) is a subset of int(*T*).
- If *A* is an open set, then *A* is a subset of *S* if and only if *A* is a subset of int(*S*).

Sometimes the second or third property above is taken as the *definition* of the topological interior.

Note that these properties are also satisfied if "interior", "subset", "union", "contained in", "largest" and "open" are replaced by "closure", "superset", "intersection", "which contains", "smallest", and "closed", respectively. For more on this matter, see interior operator below.

EXAMPLES



a is an interior point of M, because there is an ε-neighbourhood of a which is a subset of M.

- In any space, the interior of the empty set is the empty set.
- In any space *X*, if , then int(*A*) is contained in *A*.
- If *X* is the Euclidean space of real numbers, then int([0, 1]) = (0, 1).
- If *X* is the Euclidean space , then the interior of the set of rational numbers is empty.
- In any Euclidean space, the interior of any finite set is the empty set.

On the set of real numbers, one can put other topologies rather than the standard one.

- If , where has the lower limit topology, then int([0, 1]) = [0, 1).
- If one considers on the topology in which every set is open, then int([0, 1]) = [0, 1].
- If one considers on the topology in which the only open sets are the empty set and itself, then int([0, 1]) is the empty set.

These examples show that the interior of a set depends upon the topology of the underlying space. The last two examples are special cases of the following.

- In any discrete space, since every set is open, every set is equal to its interior.

- In any indiscrete space X, since the only open sets are the empty set and X itself, we have int(X) = X and for every proper subset A of X, int(A) is the empty set.

5.3.3 Interior Operator

The interior operator ^{o} is dual to the closure operator $^{-}$, in the sense that

,

and also, where X is the topological space containing S, and the backslash refers to the set-theoretic difference.

Therefore, the abstract theory of closure operators and the Kuratowski closure axioms can be easily translated into the language of interior operators, by replacing sets with their complements.

5.4 EXTERIOR OF A SET

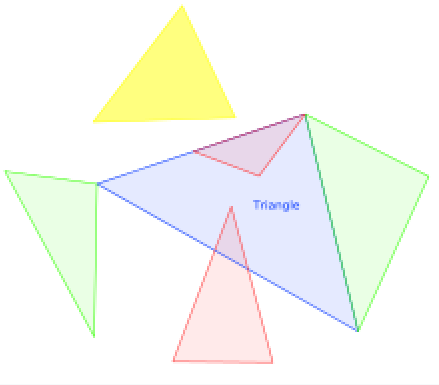
The **exterior** of a subset S of a topological space X, denoted ext(S) or Ext(S), is the interior int($X \setminus S$) of its relative complement. Alternatively, it can be defined as $X \setminus S^{-}$, the complement of the closure of S. Many properties follow in a straightforward way from those of the interior operator, such as the following.

- ext(S) is an open set that is disjoint with S.
- ext(S) is the union of all open sets that are disjoint with S.
- ext(S) is the largest open set that is disjoint with S.
- If S is a subset of T, then ext(S) is a superset of ext(T).

Unlike the interior operator, ext is not idempotent, but the following holds:

- ext(ext(S)) is a superset of int(S).

5.4.1 Interior-Disjoint Shapes



The red shapes are not interior-disjoint with the blue Triangle. The green and the yellow shapes are interior-disjoint with the blue Triangle, but only the yellow shape is entirely disjoint from the blue Triangle.

Two shapes *a* and *b* are called *interior-disjoint* if the intersection of their interiors is empty. Interior-disjoint shapes may or may not intersect in their boundary.

5.5 BOUNDARY (TOPOLOGY)

A set (in light blue) and its boundary (in dark blue).

In topology and mathematics in general, the boundary of a subset *S* of a topological space *X* is the set of points which can be approached both from *S* and from the outside of *S*. More precisely, it is the set of points in the closure of *S* not belonging to the interior of *S*. An element of the boundary of *S* is called a boundary point of *S*. The term boundary operation refers to finding or taking the boundary of a set. Notations used for boundary of a set *S* include bd(*S*), fr(*S*), and ∂S. Some authors (for example Willard, in *General Topology*) use the term frontier instead of boundary in an attempt to avoid confusion with the concept of boundary used in algebraic topology and manifold theory.[*further explanation needed*] Despite widespread acceptance of the meaning of the terms boundary and frontier, they have sometimes been used to refer to other sets. For example, the term frontier has been used to describe the residue of *S*, namely $\overline{S} \setminus S$ (the set of boundary points not in *S*).[*citation needed*] Hausdorff[1] named the intersection of *S* with its boundary the border of S (the term boundary is used to refer to this set in *Metric Spaces* by E.T. Copson).

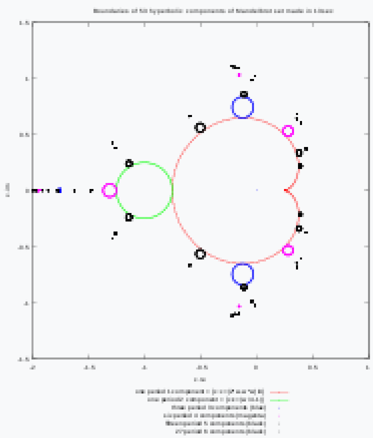
A connected component of the boundary of S is called a boundary component of S.

If the set consists of isolated points only, then the set has only a boundary and no interior.

There are several common (and equivalent) definitions to the boundary of a subset S of a topological space X:

- the closure of S without the interior of S: $\partial S = \bar{S} \setminus S^{o}$.
- the intersection of the closure of S with the closure of its complement: $\partial S = S \cap (X \setminus S)$.
- the set of points p of X such that every neighborhood of p contains at least one point of S and at least one point not of S.

Examples



Boundary of hyperbolic components of Mandelbrot set

Consider the real line **R** with the usual topology (i.e. the topology whose basis sets are open intervals). One has

- $\partial(0,5) = \partial[0,5) = \partial(0,5] = \partial[0,5] = \{0,5\}$
- $\partial\emptyset = \emptyset$
- $\partial\mathbf{Q} = \mathbf{R}$
- $\partial(\mathbf{Q} \cap [0,1]) = [0,1]$

These last two examples illustrate the fact that the boundary of a dense set with empty interior is its closure.

In the space of rational numbers with the usual topology (the subspace topology of **R**), the boundary of , where *a* is irrational, is empty.

The boundary of a set is a topological notion and may change if one changes the topology. For example, given the usual topology on $\mathbf{R}^2$, the boundary of a closed disk $\Omega = \{(x,y) \mid x^2 + y^2 \leq 1\}$ is the disk's surrounding circle: $\partial\Omega = \{(x,y) \mid x^2 + y^2 = 1\}$. If the disk is viewed as a set in $\mathbf{R}^3$ with its own usual topology, i.e. $\Omega = \{(x,y,0) \mid x^2 + y^2 \leq 1\}$, then the boundary of the disk is the disk itself: $\partial\Omega = \Omega$. If the disk is viewed as its own topological space (with the subspace topology of $\mathbf{R}^2$), then the boundary of the disk is empty.

Properties

- The boundary of a set is closed.[2]
- The boundary of the interior of a set as well as the boundary of the closure of a set are both contained in the boundary of the set.
- A set is the boundary of some open set if and only if it is closed and nowhere dense.
- The boundary of a set is the boundary of the complement of the set: $\partial S = \partial(S^C)$.
- The interior of the boundary of a closed set is the empty set.

Hence:

- *p* is a boundary point of a set if and only if every neighborhood of *p* contains at least one point in the set and at least one point not in the set.
- A set is closed if and only if it contains its boundary, and open if and only if it is disjoint from its boundary.
- The closure of a set equals the union of the set with its boundary. $\bar{S} = S \cup \partial S$.
- The boundary of a set is empty if and only if the set is both closed and open (that is, a clopen set).
- The interior of the boundary of the closure of a set is the empty set.

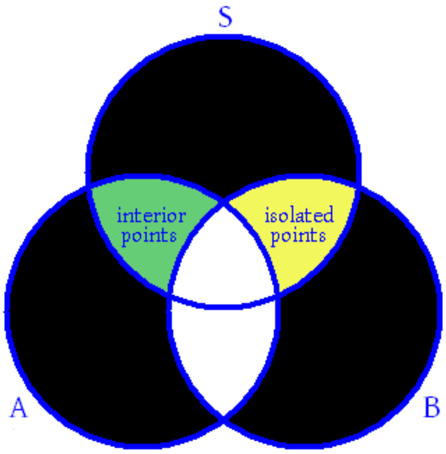


Figure 5.1

Conceptual Venn diagram showing the relationships among different points of a subset S of R^n. A = set of limit points of S, B = set of boundary points of S, area shaded green = set of interior points of S, area shaded yellow = set of isolated points of S, areas shaded black = empty sets. Every point of S is either an interior point or a boundary point. Also, every point of S is either an accumulation point or an isolated point. Likewise, every boundary point of S is either an accumulation point or an isolated point. Isolated points are always boundary points.

5.5.1 Boundary of a Boundary

For any set S, $\partial S \supseteq \partial\partial S$, with equality holding if and only if the boundary of S has no interior points, which will be the case for example if S is either closed or open. Since the boundary of a set is closed, $\partial\partial S = \partial\partial\partial S$ for any set S. The boundary operator thus satisfies a weakened kind of idempotence.

In discussing boundaries of manifolds or simplexes and their simplicial complexes, one often meets the assertion that the boundary of the boundary is always empty. Indeed, the construction of the singular homologyrests critically on this fact. The explanation for the apparent incongruity is that the topological boundary (the subject of this article) is a slightly different concept from the boundary of a manifold or of a simplicial complex. For example, the boundary of an open disk viewed as a manifold is empty, while its boundary in the sense of topological space is the circle surrounding the disk.

5.6 SUBSPACE TOPOLOGY

In topology and related areas of mathematics, a subspace of a topological space *X* is a subset *S* of *X* which is equipped with a topology induced from that of *X* called the subspace topology (or the relative topology, or the induced topology, or the trace topology).

GIVEN A TOPOLOGICAL SPACE AND A SUBSET OF ,
THE SUBSPACE TOPOLOGY **ON IS DEFINED BY**

That is, a subset of is open in the subspace topology if and only if it is the intersection of with an open set in . If is equipped with the subspace topology then it is a topological space in its own right, and is called a **subspace** of . Subsets of topological spaces are usually assumed to be equipped with the subspace topology unless otherwise stated.

Alternatively we can define the subspace topology for a subset of as

the coarsest topology for which the inclusion map is continuous.

More generally, suppose is an injection from a set to a topological space . Then the subspace topology is defined as the coarsest topology for which is continuous. The open sets in this topology are precisely the ones of the form for open in . is then homeomorphic to its image in (also with the subspace topology) and is called a topological embedding.

A subspace is called an **open subspace** if the injection is an open map, i.e., if the forward image of an open set of is open in . Likewise it is called a **closed subspace** if the injection is a closed map.

5.6.1 Terminology

The distinction between a set and a topological space is often blurred notationally, for convenience, which can be a source of confusion when one first encounters these definitions. Thus, whenever is a subset of , and is a topological space, then the unadorned symbols "" and "" can often be used to refer both to and considered as two subsets of , and also to and as the topological spaces, related as discussed above. So phrases such as " an open subspace of " are used to mean that is an open subspace of , in the sense used below -- that is that: (i) ; and (ii) is considered to be endowed with the subspace topology.

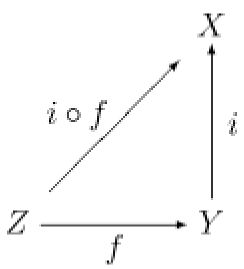
EXAMPLES

In the following, represents the real numbers with their usual topology.

- The subspace topology of the natural numbers, as a subspace of , is the discrete topology.
- The rational numbers considered as a subspace of do not have the discrete topology ({0} for example is not an open set in). If *a* and *b* are rational, then the intervals (*a*, *b*) and [*a*, *b*] are respectively open and closed, but if *a* and *b* are irrational, then the set of all rational *x* with *a* < *x* < *b* is both open and closed.
- The set [0,1] as a subspace of is both open and closed, whereas as a subset of it is only closed.
- As a subspace of R_1 , [0, 1] ∪ [2, 3] is composed of two disjoint *open* subsets (which happen also to be closed), and is therefore a disconnected space.
- Let *S* = [0, 1) be a subspace of the real line . Then [0, 1/2) is open in *S* but not in . Likewise [½, 1) is closed in *S* but not in R . *S* is both open and closed as a subset of itself but not as a subset of R.

PROPERTIES

The subspace topology has the following characteristic property. Let be a subspace of and let be the inclusion map. Then for any topological space a map is continuous if and only if the composite map is continuous.



This property is characteristic in the sense that it can be used to define the subspace topology on .

We list some further properties of the subspace topology. In the following let be a subspace of .

- If is continuous the restriction to is continuous.
- The closed sets in are precisely the intersections of with closed sets

  in .
- If is a subspace of then is also a subspace of with the same topology. In other words the subspace topology that inherits from is the same as the one it inherits from .
  - Suppose is an open subspace of (so). Then a subset of is open in if and only if it is open in .
  - Suppose is a closed subspace of (so). Then a subset of is closed in if and only if it is closed in .
  - If is a basis for then is a basis for .
  - The topology induced on a subset of a metric space by restricting the metric to this subset coincides with subspace topology for this subset.

5.6.2 The Subspace Topology

We now consider some ways of getting new topologies from old ones.

Definition

If A is a subset of a topological space $(X, \mathcal{T}_X)$, we define the **subspace topology** $\mathcal{T}_A$ on A by:

$B \in \mathcal{T}_A$ if $B = A \cap C$ for some $C \in \mathcal{T}_X$.

Examples

1. Restricting the metric on a metric space to a subset gives this topology.
   For example, On $X = [0, 1]$ with the usual metric inherited from **R**, the open sets are the intersection of [0, 1] with open sets of **R**.

So, for instance, $[1, {}^{1}/_{4}) = (-1, {}^{1}/_{4}) \cap [0, 1]$ and so is an open subset of the *subspace* X.

Remark

Note that as in this example, sets which are open in the subspace are not necessarily open in the "big space".

2. The subspace topology on $\mathbf{Z} \subset \mathbf{R}$ (with its usual topology/metric) is the discrete topology.
3. The subspace topology on the x-axis as a subset of $\mathbf{R}^2$ (with its usual topology) is the usual topology on $\mathbf{R}$.

Remark

If we take the inclusion map $i: A \to X$ then the subspace topology is the *weakest* topology (*fewest* open sets) on A in which this map is continuous.

Proof

If $B \subset X$ is open then $i^{-1}(B) = A \cap B$.

5.6.3 The Product Topology

Given topological spaces X and Y we want to get an appropriate topology on the Cartesian product $X \times Y$.

Obvious method

Call a subset of $X \times Y$ open if it is of the form $A \times B$ with A open in X and B open in Y.

Difficulty

Taking $X = Y = \mathbf{R}$ would give the "open rectangles" in $\mathbf{R}^2$ as the open sets. These subsets are open, but unfortunately there are lots of other sets which are open too.

We are therefore forced to work a bit differently.

Definition

A set of subsets $\mathcal{B}$ is a **basis** of a topology $\mathcal{T}$ if every open set in $\mathcal{T}$ is a union of sets of $\mathcal{B}$.

Example

In any metric space the set $\mathcal{B}$ of all ε-neighbourhoods (for all different values of ε) is a basis for the topology.

Remark

This is a very helpful concept. For example, to check that a function is continuous you need only verify that $f^{-1}(B)$ is open for all sets B in a basis -- usually much smaller than the whole collection of open sets.

We can now define the topology on the product.

Definition

If X and Y are topological spaces, the **product topology** on $X \times Y$ is the topology whose *basis* is $\{A \times B \mid A \in \mathcal{T}_X, B \in \mathcal{T}_Y\}$.

Examples

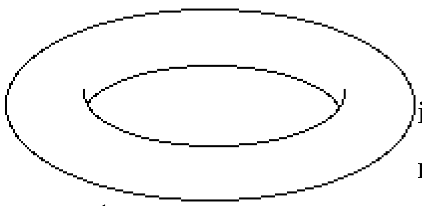
1. The topology on $\mathbf{R}^2$ as a product of the usual topologies on the copies of $\mathbf{R}$ is the usual topology (obtained from, say, the metric d_2).

   Proof

   The sets of the basis are open rectangles, and an ε-neighbouhood U in the metric d_2 is a disc. It is easy to see that every point of U can be contained in a small open rectangle lying inside the disc. Hence U is a union of (infinitely many!) of these rectangles and hence is in the product topology.

   Since every open set in the d_2 metric is a union of ε-neighbourhoods, every open set can be written as a union of the open rectangles.

   □

2. torus is the surface  in $\mathbf{R}^3$:
   It can also be regarded as the product $S^1 \times S^1$ where S^1 is a circle (the curve, not the interior) in $\mathbf{R}^2$. In this way it can be thought of as a subset of $\mathbf{R}^4$. The topology on S^1 is the subspace topology as a subset of $\mathbf{R}^2$ and

so we get the product topology on $S^1 \times S^1$.

Fortunately this is the same as the topology on the torus thought of as a subset of $\mathbf{R}^3$.

Proof

A basis for the subspace topology on S^1 is the set of "arcs"

Hence a basis for the product topology on $S^1 \times S^1$ is sets of the form:

A basis for the subspace topology on the torus as a subset of $\mathbf{R}^3$ is the intersection of the torus with ε-neighbourhoods of $\mathbf{R}^3$ (which are "small balls") and hence are sets of the form:

As before, one can get these "ovals" as unions of the small "bent rectangles".

□

3. Take the topology $\mathcal{T} = \{ \varnothing, \{a, b\}, \{a\} \}$ on $X = \{a, b\}$.
   Then the product topology on $X \times X$ is $\{ \varnothing, X \times X, \{(a, a)\}, \{(a, a), (a, b)\}, \{(a, a), (b, a)\}, \{(a, a), (a, b), (b, a)\} \}$ where the last open set in the list is *not* in the basis.

Remark

Given any product of sets $X \times Y$, there are *projection maps* $\square_X$ and $\square_Y$ from $X \times Y$ to X and to Y given by $(x, y) \mapsto x$ and $(x, y) \mapsto y$.

The product topology on $X \times Y$ is the weakest topology (*fewest* open sets) for which both these maps are continuous.

5.6.4 The Identification Topology

Recall that an *equivalence relation* ~ (a reflexive, symmetric, transitive relation) on a set X can be regarded as a method of partitioning X into disjoint subsets (called *equivalence classes*).

We shall denote the set of equivalence classes by $X/\sim$.

The *identification* or *quotient topology* gives a method of getting a topology on $X/\sim$ from a topology on X.

To see why we would want to do something like this, we will look at some examples.

Examples

1. Let X be the closed unit interval $[0, 1] \subset \mathbb{R}$.
   Define an equvalence relation on X by $x \sim y$ if and only if $x = y$ or $\{x, y\} = \{0, 1\}$.
   Then every point is its own equivalence class, except for $\{0, 1\}$ which forms one class.
   So this relation has the effect of "identifying" 0 and 1 and leaving everything else alone.

   What would we *like* the topology to look like?

2. Let $X = \mathbb{R}$ and define $\sim$ by $x \sim y$ if and only if $x - y \in \mathbb{Z}$.
   Then $X/\sim$ is the set of cosets $\mathbb{R}/\mathbb{Z}$ of the additive group.
   What would we like the topology to be?
   A clue comes from group theory.
   Define $f: \mathbb{R} \to \mathbb{C} - \{0\}$ by $z \mapsto \exp(2\pi it)$. Then $ker(f) = \mathbb{Z}$ and the image of f is the unit circle in the C-plane.
   Thus, by the First Isomorphism Theorem of group theory, $\mathbb{R}/\mathbb{Z} \cong$ a circle and it would be a good idea to consider it with its subspace topology inherited from the plane.

In fact these two examples are the same. To see this, take as a representative of each equivalence class ($\equiv$ coset) an element in $[0, 1)$.
Then every equivalence class has a unique representative, but one should arrange things so that classes with representatives close to 1 should be near the one with representative 0.
So we fix the topology so that this happens.

If we have an equivalence relation $\sim$ on X we get a natural projection map $\pi: X \to X/\sim$ got by mapping each point to its equivalence class.

Definition

Notes

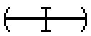
The identification topology on *X*/~ is defined by:
A set $A \subset X/\sim$ is open if and only if □ $^{-1}(A)$ is open in *X*.

Remarks

1. This topology is the strongest (≡ *most* open sets) in which □ is continuous.
2. Note that □ $^{-1}(A)$ consists of all those points whose equivalence classes are in *A*.

Look again at the previous examples

We have $X = [0, 1]$ and □ : $X \to X/\sim$ identifying the end-points of the interval. If *U* is a set of *X*/~ containing the equivalence class {0, 1} then □ $^{-1}(U)$ contains both 0 and 1 and hence if it is open contains a set like 

So small open sets of *X*/~ (ε-neighbourhoods) are "the same" as those of *X* except "at the end-points" where they look like the sets  made by "gluing together two bits". It is easy to see that this is the same as the subspace topology on the circle as a subset off the plane.

Topologists like to consider spaces made by "shrinking a subset to a point".

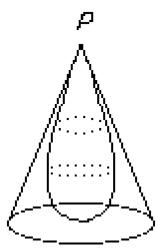
Notation

If *X* is a topological space and *A* is a subset of *X*, we denote by *X*/*A* the equivalence classes *X*/~ under the relation $x \sim y$ if $x = y$ or $x, y \in A$.
So for $x \notin A$, $\{x\}$ is an equivalence class and *A* is a single class.

Examples

1. The above example is [0, 1]/{0, 1}
2. Let *X* be the closed unit disc and let *A* be the "boundary circle". Then *X*/*A* is homeomorphic to the unit sphere $S^2 \subset \mathrm{R}^3$.

   Proof



Look at the "teardrop" shown and "stereoscopic projection from the "point" P to a horizontal plane. The inverse image of an open set which contains the whole boundary is then an open set which contains P. Every other point is mapped in a one-one way. It is easy to verify that this map is a homeomorphism.

□

Check In Progress-II

Q 1. Define Product Topology.

Solution :

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

Q. 2 Define subspce topology.

Solution .

…………………………………………………………………………

…………………………………………………………………………

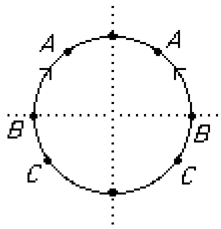
…………………………………………………………………………

…………………………………………………………………………

5.6.6 More Identification Spaces

A sphere (again)

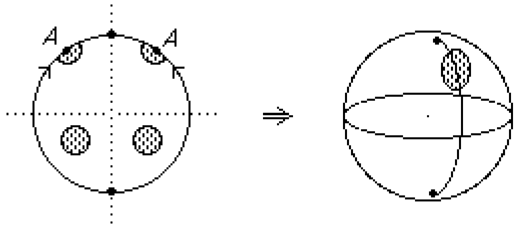
Start with a disc. Use an equivalence relation to identify points on the boundary as shown.



That is, $(x_1, y_1) \sim (x_2, y_2)$ if $(x_1, y_1) = (x_2, y_2)$ or $x_1 = -x_2$ and $y_1 = y_2$ and $x_1^2 + y_1^2 = 1$ and $x_2^2 + y_2^2 = 1$.

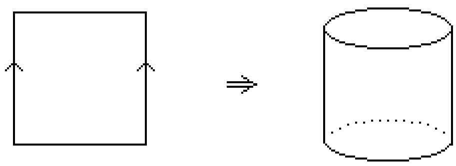
This is like folding a piece of pastry to make a "bridie" or "pastie"

A inverse image of a neighbourhood of a point in X/~ is like an ordinary neighbourhood for an interior point and is the union of a pair of "semi-discs" for a point on the boundary.



A cylinder

Glue the ends of a strip using the same equivalence relation used to make the circle from a closed interval.

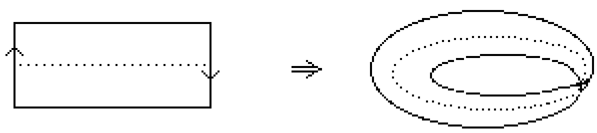


This space then gets the same topology as a subset of R^3, as a product $S^1 \times I$ and as an identification space.

A Möbius band

This was invented by the German mathematician August Möbius (1790 to 1868) in 1858.

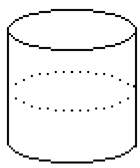
"Glue" a strip as above, but this time after a half twist.



Note that the "centre line" becomes a circle in R^3.

Remark

One can make a cylinder by giving the strip a *full* twist before gluing. This produces something homeomorphic to the above cylinder but with an embedding in R^3 in a very different way.



You can see this by cutting the surface along the dotted line shown.

5.6.7 Separation Axioms

We saw earlier how the ideas of convergence could be interpreted in a topological rather than a metric space:

A sequence (a_i) converges to α if every open set containing α contains all but a finite number of the $\{a_i\}$.

Unfortunately, this definition does not give some of thr "nice" properties we get in a metric space.

For example, if a sequence in a metric space converges, it has a unique limit, but in a topological space this need not happen. For example, in R with the trivial topology every sequence converges to every point.

To recove the nice properties of convergence we need to have "enough" open sets in the topology. Topologists have devised various *separation axioms* to classify how this happens.

Definition

A topological space X is called Hausdorff if every pair of points can be separated by open sets.

That is, if $x_1 \neq x_2 \in X$ then there are *disjoint* open sets U_1 and U_2 with $x_1 \in U_1$ and $x_2 \in U_2$.

Remarks

1. Felix Hausdorff (1869 to 1942) introduced this idea. He was also responsible for the first formulation of the idea of fractional dimension encountered in fractal Geometry.
2. The Hausdorff condition is sometimes called T_2 . This axiom is one of a number of separation axioms: T_0 , T_1 , T_2 , T_3 , $T_{3^1/2}$, T_4 . These were named by Heinreich Tietze (1880 to 1964) in 1923. The T stands for *trennung* (= *separation* in German). Some references call them the *Tychonoff axioms* after Andrei Tychonoff (1906 to 1993).
   Just for the record, T_0 spaces are sometimes called Kolmogorov spaces, T_1 spaces are called Fréchet spaces, T_2 spaces are Hausdorff, T_3 spaces are *regular*, $T_{3^1/2}$ spaces are *completely regular*. We will see

about T_4 spaces shortly.

See this link for more details.

3. In a Hausdorff space, distinct points are "housed off" from one another by open sets.

Remark

It follows that every finite set is closed in a Hausdorff space and the topology is therefore *stronger* than the cofinite topology.

The other separation axiom we will consider is:

Definition

A topological space X is called **normal** if every disjoint pair of closed sets can be separated by open sets.
That is, if A_1 *and* A_2 are disjoint closed subsets of X then there are *disjoint* open sets U_1 and U_2 with $A_1 \subset U_1$ and $A_2 \subset U_2$.

Remark

If every point is a closed set (that is T_1) then such a normal space is Hausdorff. [normal + T_1 = T_4]

Remark

Note that the distance between disjoint closed sets *may* be 0 (but they can still be separated by open sets).

Examples

1. As above, all metric spaces are both Hausdorff and normal.
2. The space $X = \{a, b\}$ with $\mathcal{T} = \{ \varnothing, X, \{a\} \}$ is *not* Hausdorff since a, b cannot be separated by open sets.
   It is, however, normal since there are no non-empty disjoint closed sets.

Remarks

1. Finding a Hausdorff space which is not normal is possible, but tricky!

2. By demanding more separation axioms one gets closer to a metric space. Paul Urysohn (1898 to 1924) proved in 1923 that any T_4 space with a *countable basis* is metrisable (that is, the topology may be obtained from a metric).
   In fact this is not a necessary condition for metrisability. For example, **R** with the discrete topology is metrisable but does not have a countable basis. Marshall Stone (1903 to 1989) and R H Bing (1914 to 1986) found a necessary and sufficient condition for metrisability in 1950.

Example Topological Subspaces

Recall from the Topological Subspaces page that if (X,τ) is a topological space and A⊆X then the subspace topology on A is defined to be:

(1)

$$\tau A=\{A\cap U:U\in\tau\}$$

We verified that τA is indeed a topology for any subset A of X.

We will now look at some examples of subspace topologies.

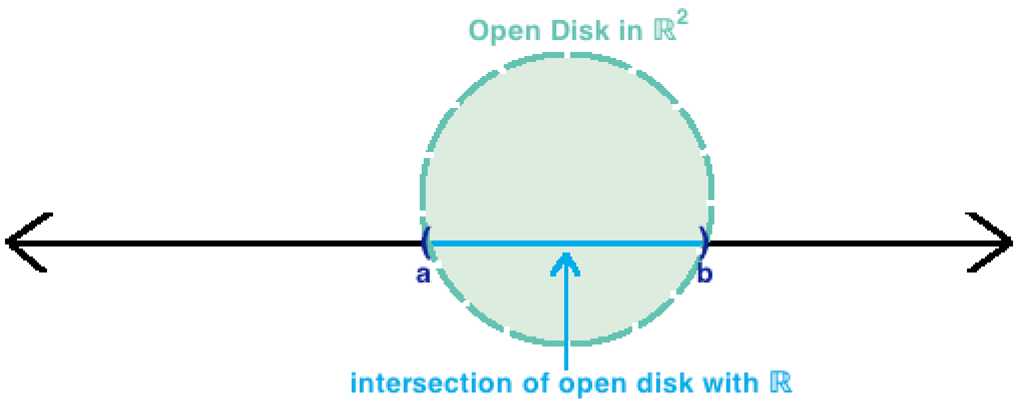
Example 1

Consider the topological space (R2,τ) **where** τ **is the usual topology of open disks in** R2**. Determine what the subspace topology is for the subset** A={(x,0)∈R2:x∈R}⊆R2**.**

Note that the set A={(x,0)∈R2:x∈R} is simply the real line R.

Geometrically we can see that the subspace topology τA will simply be the usual topology on R.

To see this, consider any open set in R with respect to the usual topology of open intervals in R. Then any open interval (a,b) can be constructed by taking an open disk in R2 that intersects the line y=0 at the points (a,0), (b,0).



Since every open set in R is a union of these open intervals, we see that the subspace topology on R is simply the usual topology on R.

Example 2

Consider the topological space (R,τ) **where** τ **is the usual topology of open intervals in** R**. Verify that the subspace topology on** Z⊆R **is the discrete topology on** Z**.**

Let x∈Z. Then the open interval (x−1/2,x+1/2)∩Z={x}. Hence every singleton set {x} is contained in the subspace topology on Z. But this implies that that τZ is the discrete topology on Z.

Basic Properties of Subspaces

The following basic property is often taken for granted.

Theorem. If (X, T) is a topological space and $Z \subseteq Y \subseteq X$ are subsets, then we can form the subspace topology on Z in two ways:

- by taking the subspace topology $T \mapsto T_Z$ from $Z \subseteq X$; and
- by taking successive subspaces $T \mapsto T_Y$ which is a topology on Y, then $T_Y \mapsto (T_Y)_Z$.

The two topologies are identical.

The proof is straightforward: in the first case, the class of all open subsets of *Z* is given by *U* ∩ *Z* for open subsets *U* of *X*; in the second case, the class is given by (*U* ∩ *Y*) ∩ *Z* = *U*∩ *Z* for open subsets *U* of *X*. They're identical. ♦

The following properties are also surprisingly useful in practice.

Theorem. *Let Y be a subspace of (X, T). If Y is open in X, then any open subset of Y is an open subset of X. If Y is closed in X, then any closed subset of Y is a closed subset of X.*

Proof.

For the first statement, an open subset of Y is of the form $V = U \cap Y$ for some open subset U of X. Since U and Y are both open in X, so is $V = U \cap Y$. The same proof holds for the second statement by replacing 'open' with 'closed'. ♦

Theorem. *Let* (X, T) *be a topological space with subspace* (Y, T_Y).

- *If B is a basis for T, then* $B_Y := \{U \cap Y : U \in B\}$ *is a basis for Y.*
- *If S is a subbasis for T, then* $S_Y := \{U \cap Y : U \in S\}$ *is a subbasis for Y.*

Proof.

For the first statement, we first verify that B_Y is indeed a basis of some topology over Y:

- $\cup_{V \in B_Y} V = \cup_{U \in B}(U \cap Y) = (\cup_{U \in B} U) \cap Y = X \cap Y = Y.$
- Any two elements of B_Y are of the form $U_1 \cap Y, U_2 \cap Y$ for some basic open subsets $U_1, U_2 \in B$. Since B is a basis, $U_1 \cap U_2 = \cup_i V_i$ for some $V_i \in B$. This gives $(U_1 \cap Y) \cap (U_2 \cap Y) = (U_1 \cap U_2) \cap Y = (\cup_i V_i) \cap Y = \cup_i (V_i \cap Y)$ which is a union of elements in B_Y.

Finally, we need to show B_Y generates the topology T_Y.

- Every element $U \cap Y \in B_Y$ (for some element U in B) is open in Y by definition. So $B_Y \subseteq T_Y$.
- Conversely, any element of T_Y is of the form $U \cap Y$ for some open subset U of X. Since B is a basis for (X, T), $U = \cup_i V_i$ for some $V_i \in B$. So $U \cap Y = \cup_i (V_i \cap Y)$ which is a union of elements in B_Y.

This concludes the proof for the first statement.

For the second, suppose the intersections of finitely many elements of *S* form basis *B*. It suffices to show that the intersections of finitely many elements of S_Y form basis B_Y.

- In one direction, note that $S \subseteq B \implies S_Y \subseteq B_Y$.
- Conversely, any element of B_Y is of the form $U \cap Y$ for some $U \in B$. So $U = V_1 \cap V_2 \cap \ldots \cap V_n$ for finitely many $V_i \in S$. But this gives $U \cap Y = \cap_i (V_i \cap Y)$ for finitely many $V_i \cap Y \in S_Y$.

5.7 SUMMARY

We study in this unit

1. Any subset of a topological space X inherits a topology from it. The inheritance is consistent across inclusion chains of topological spaces.
2. With spaces and subspaces, one should be more careful when talking about "open sets", i.e. mention what it's open in.
3. If Y is open (resp. closed) in X and Z is open (resp. closed) in Y, then Z is open (resp. closed) in X.
4. If Y is a subspace of X, then a basis (resp. subbasis) of X restrict to give a basis (resp. subbasis) of Y.

5.8 KEYWORD

Discrete : Individually separate and distinct

Mobius Band : A ***Möbius strip***, ***Möbius band***, or Möbius loop also spelled Mobius or Moebius

Terminology : The body of terms used with a particular technical application in a subject of study, profession, etc

5.9 EXERCISE

1. A subspace of (*X*, *d*) with the discrete metric is still discrete.

2. Pick the half-open interval $Y = [0, 1) \subset \mathbf{R}$
   . Then $[0, 1/2) = (-1, 1/2) \cap Y$ is open in *Y* but not open in **R**.
3. Consider the subset $Y = (-\infty, -1] \cup \{0\} \cup [1, \infty)$ of the real line **R**. The singleton set {0} is an open subset of *Y* since {0} = *N*(0, 1/2). Furthermore, it is closed since the complement is a union of two open subsets.
4. Consider the subset **Z** of **R** under the usual metric. Then the resulting subspace is the discrete space even though the induced metric *d*(*m*, *n*) = |*m*–*n*| is not exactly the discrete metric.
5. Let $X = \mathbf{R}^2$ and *Y* be the set of points (*x*, *y*) satisfying $x^2 + y^2 = 1$. Geometrically, *Y* is a circle. Here, we'll think of it as a topological space with the subspace topology inherited from *X*. The space is denoted S^1. More generally, for each positive integer *n*, the space S^n is the subspace of $\mathbf{R}^{n+1}$ comprising of all points $(x_1, x_2, \ldots, x_{n+1})$ satisfying $\sum_{i=1}^{n+1} x_i^2 = 1.$
6. Consider *X* = **N** under the right order topology.
   A. If *Y* = {1, 2, 3}, then the subspace topology gives { (empty set), {3}, {2, 3}, *Y* }.
   B. If *Y* is the set of even numbers, then the bijection $X \to Y, n \mapsto 2n$ preserves the structure of topological spaces.

5.10 ANSWER FOR CHECK IN PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 1

Q 2 Check in Section 1

Check in Progress-II

Answer Q. 1 Check in Section 6

Q 2 Check in Section 5

5.11 SUGGESTION READING AND REFERENCES

Kuratowski, Kazimierz *(1922).* "Sur l'Operation Ā de l'Analysis Situs" (PDF)*. Fundamenta Mathematicae. Warsaw: Polish Academy of Sciences. 3: 182–199.* ISSN 0016-2736.

Hausdorff, Felix (1914). Grundzüge der Mengenlehre. Leipzig: Veit. p. 214. ISBN 978-0-8284-0061-9. Reprinted by Chelsea in 1949.

^ Mendelson, Bert (1990) [1975]. Introduction to Topology (Third ed.). Dover. p. 86. ISBN 0-486-66352-3. Corollary 4.15 For each subset A, Brdy (A) is closed.

- Bourbaki, Nicolas, Elements of Mathematics: General Topology, Addison-Wesley (1966)
- Steen, Lynn Arthur; Seebach, J. Arthur Jr. (1995) [1978], Counterexamples in Topology (Dover reprint of 1978 ed.), Berlin, New York: Springer-Verlag, ISBN 978-0-486-68735-3, MR 0507446
- Willard, Stephen. General Topology, Dover Publications (2004) ISBN 0-486-43479-6

UNIT 6 TOPIC: CONTINUOUS MAPPING

STRUCTURE

6.0 OBJECTIVE

- Learn Closed and Open Mapping
- Learn perfect and quotient mapping
- Learn Stone-Weiertrass Theorem
- Learn Infinite Product Theorem
- Learn Homomorphism and Preparation theorem

6.1 INTRODUCTION

A continuous map is a continuous function between two topological spaces. In some fields of mathematics, the term "function" is reserved for functions which are into the real or complex numbers. The word "map" is then used for more general objects.

A map $F : X \to Y$ is continuous iff the preimage of any open set is open.

6.1.1 Continuous Mapping

A mapping $f : X \to Y$ from a topological space X into a topological space Y such that for every point $x_0 \in X$ and for every neighbourhood $V = V(f(x_0))$ of its image $f(x_0)$ there is a neighbourhood $U = U(x_0)$ of x_0 such the $f(U) \subset V$. This definition is a rephrasing of the neighbourhood definition of continuity of a function of a real variable (see Continuous function). There are many equivalent definitions of continuity. Thus, for the continuity of a mapping $f : X \to Y$ it is necessary and sufficient that any one of the following conditions holds:

a) the inverse image $f^{-1}(G)$ of every open set G in Y is open in X;

b) the inverse image $f^{-1}(F)$ of every closed set F in Y is closed in X;

c) $f(\bar{A}) \subset \overline{f(A)}$ for every set $A \subset X$ (the image of the closure is contained in the closure of the image).

The concept of a continuous function, which was correctly stated already by B. Bolzano and A.L. Cauchy, played an important role in the mathematics of the 19th century. Weierstrass' function, which is nowhere differentiable, "Cantor's staircase" and Peano's curve pointed to the need of considering more special cases of continuity. The necessity of selecting special classes of mappings became even more urgent when continuous mappings of more general objects — topological spaces — were considered. One could mention the following important types of continuous mappings: topological mappings or homeomorphisms, perfect mappings, closed mappings, open mappings, and quotient mappings (cf. Homeomorphism; Closed mapping; Open mapping; Perfect mapping; Quotient mapping). If $f:X\to Y$ and $g:Y\to Z$ are two continuous mappings, then their composite $g\circ f:X\to Z$ that is, the mapping $X\to^{f} Y\to^{g} Z$ is also continuous. Every identity mapping $I_X:X\to X$ is evidently continuous. Therefore, topological spaces and continuous mappings form a category.

One of the aims of topology is the classification of both spaces and mappings. Its essence consists of the following: three fundamental and closely connected problems are selected. 1) In what case can every space of a certain fixed class $\mathcal{A}$ be mapped into some space of a class $\mathcal{B}$ by a continuous mapping belonging to a class $\mathcal{L}$ 2) By what intrinsic properties can one characterize spaces belonging to the class MC which contains the images of spaces in a class C under continuous mappings in a class M 3) Let $\mathrm{Hom}\ (\mathcal{A},\mathcal{B})$ be the set of continuous mappings whose domains of definition are spaces in a class $\mathcal{A}$ and whose ranges are spaces in a class $\mathcal{B}$, and let $\mathcal{L}$ be some other class of mappings. What are the properties of mappings of the class $\mathrm{Hom}\ (\mathcal{A},\mathcal{B})\cap\mathcal{L}$

These general statements comprise in particular, the following question: What topological properties are preserved by mappings of one kind or another under transition from a space to its image or inverse image? 1) Every n-dimensional space (in the sense of $\dim$) can be mapped essentially (see Essential mapping) onto an n-dimensional cube. 2) A pointwise-countable base is preserved under perfect (even under bifactorial) mappings. 3) Every closed mapping f in the class $\mathrm{Hom}\ (\mathcal{A},\mathcal{B})$, where $\mathcal{A}$ is the class of zero-dimensional spaces with a countable base and $\mathcal{B}$ is the class of n-dimensional spaces with a countable base, is at least $(n+1)$-fold.

The first specific problems of this kind were solved at the beginning of the 20th century. Such are, for example, the representation of an arbitrary compactum as a continuous image of the Cantor perfect set (Aleksandrov's theorem); the characterization of metric spaces with a countable base as open continuous images of subspaces of the space of irrational numbers (Hausdorff's theorem); the description of locally connected continua as continuous images of an interval. The solution of these problem not only made it possible to answer questions about interrelationships between previously known spaces, but also led to the emergence of interesting new classes of spaces. Such are, for example, dyadic compacta, paracompact p-spaces, perfect n-dimensional spaces, and pseudo-compact spaces.

The concept of a real-valued continuous function, that is, a continuous mapping of a topological space into $\mathbf{R}$, which lies at the basis of the theory of functions, plays an important role also in general topology. Here one must mention first of all the Urysohn lemma, the Brouwer–Urysohn theorem on the extension of continuous functions from closed subsets of normal spaces, A.N. Tikhonov's definition of completely-regular spaces (cf. Completely-regular space), and the Stone–Weierstrass theorem. These and other investigations led to the creation of a theory of rings of continuous functions, the methods of which turned out to be quite fruitful in general topology.

A substantial part of dimension theory is the study of the behaviour of dimensional characterizations of spaces under transition to an image or inverse image by mappings of one class or another. Here an important role is played by ϵ-shifts, ϵ-mappings, ω-mappings, essential mappings, finite-to-multiple mappings, countable-to-multiple mappings, zero-dimensional mappings, n-dimensional mappings, etc. Here the method of continuous mappings leads to a mutual enrichment of and interrelations between domains of general topology of totally different origin, such as dimension theory, which has an intuitive geometric meaning, and the theory of cardinal invariants, which is abstract in character.

One of the characteristics of dimension is the possibility of extending a continuous mapping from a closed subset to an n-dimensional sphere. This is one of the versions of the theorem on the extension of mappings, which, like the fixed-point theorem closely connected with it, is of prime importance in branches of modern mathematics such as topology, algebra, function theory, functional analysis, and differential equations.

One of the best studied classes of continuous mappings is that of the perfect irreducible mappings (cf. Perfect irreducible mapping). The theorem on the absolute of a regular space stimulated entire series of investigations in this domain. In particular, the concept of an absolute has been extended to the class of all Hausdorff spaces. Closely connected with the concept of a continuous mapping turned out to be that of a θ-proximity, which made it possible to given an intrinsic description of all perfect continuous inverse images of an arbitrary compactum. The extension of the theory of irreducible continuous mappings to the class of all Hausdorff spaces showed that continuous mappings are insufficient for the study of non-regular spaces and that is more natural to consider θ-continuous mappings in this case.

The selection of uniformly-continuous functions from the class of all numerical functions of one or several variables became one of the starting points of research leading to the notion of a uniform topology.

Continuous mappings of one type or another lie at the basis of the theory of retracts, splines and homology theory. A major role in modern mathematics is played by various aspects of the theory of multi-valued mappings (cf. Multi-valued mapping). Questions related to continuous mappings of Euclidean spaces are interesting by the wealth of ideas they contain.

A basic concept in mathematical analysis.

Let f be a real-valued function defined on a subset E of the real numbers $\mathbf{R}$, that is, $f:E\rightarrow\mathbf{R}$. Then f is said to be continuous at a point $x_0\in E$ (or, in more detail, continuous at x_0 with respect to E) if for any $\epsilon>0$ there exists a $\delta>0$ such that for all $\mathbf{x}\in E$ with $|\mathbf{x}-x_0|<\delta$ the inequality

$$|f(x)-f(x_0)|<\epsilon$$

is valid. If one denotes by

$$U(x_0,\delta)=(x_0-\delta,x_0+\delta)$$

and

$$V(f(x_0),\epsilon)=(f(x_0)-\epsilon,f(x_0)+\epsilon)$$

the δ- and ϵ-neighbourhoods of x_0 and $f(x_0)$, respectively, then the definition above can be rephrased as follows: f is called continuous at a point $x_0\in E$ if for each ϵ -

Notes

neighbourhood $V = V(f(x_0), \epsilon)$ of $f(x_0)$ there is a δ -neighbourhood $U = U(x_0, \delta)$ of x_0 such that $f(U \cap E) \subset V$.

By using the concept of a limit one can say that f is continuous at a point x_0 if its limit with respect to the set E exists at that point and if this limit is equal to $f(x_0)$:

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = f(x_0).$$

This is equivalent to

$$\lim_{\substack{\Delta x \to 0 \\ x \in E}} \Delta y = 0,$$

where $\Delta x = x - x_0$, $x \in E$, and $\Delta y = f(x) - f(x_0)$; that is, to an infinitely small increment of the argument at x_0 corresponds an infinitely small increment of the function.

In terms of the limit of a sequence, the definition of continuity of a function at x_0 is: f is continuous at x_0 if for every sequence of points $x_n \in E$, $n = 1, 2, \ldots$, for which $\lim_{n \to \infty} x_n = x_0$, one has

$$\lim_{n \to \infty} f(x_n) = f(x_0).$$

All these definitions of a function being continuous at a point are equivalent.

If f is continuous at x_0 with respect to the set $E \cap \{x : x \geq x_0\}$ (or $E \cap \{x : x \leq x_0\}$), then f is said to be continuous on the right (or left) at x_0.

All basic elementary functions are continuous at all points of their domains of definition. An important property of continuous functions is that their class is closed under the arithmetic operations and under composition of functions. More accurately, if two real-valued functions $f : E \to \mathbf{R}$ and $g : E \to \mathbf{R}$, $E \subset \mathbf{R}$, are continuous at $x_0 \in E$, then so is their sum $f + g$, difference $f - g$ and product fg, and when $g(x_0) \neq 0$, also their quotient f/g (which is necessarily defined in the intersection of E with a certain neighbourhood of x_0). If, as before, $f : E \to \mathbf{R}$ is continuous at $x_0 \in E$ and $\phi : D \to \mathbf{R}$, $D \subset \mathbf{R}$, is such that $\phi(D) \subset E$, so that the composite $f \circ \phi$ makes sense, if there is a $t_0 \in D$ such that $\phi(t_0) = x_0$ and if ϕ is continuous at t_0, then $f \circ \phi$ is also continuous at t_0. Thus, in this case

$$\lim_{t \to t_0} f[\phi(t)] = f\left[\lim_{t \to t_0} \phi(t)\right] = f[\phi(t_0)],$$

that is, in this sense the operation of limit transition commutes with the operation of applying a continuous function. From these properties of continuous functions it follows that not only the basic, but also arbitrary elementary functions are continuous in their domains of definition. The property of continuity is also preserved under a uniform limit transition: If a sequence of functions $\{f_n\}$ converges uniformly on a set E and if every f_n is continuous at $x_0 \in E$, $n = 1, 2, \ldots$, then

$$f = \lim_{n \to \infty} f_n$$

is continuous at x_0.

If a function $f : E \to \mathbf{R}$ is continuous at every point of E, then f is said to be continuous on the set E. If $x_0 \in E_1 \subset E$ and f is continuous at x_0, then the restriction of f to E_1 is also continuous at x_0. The converse is not true, in general. For example, the restriction of the Dirichlet function either to the set of rational numbers or to the set of irrational numbers is continuous, but the Dirichlet function itself is discontinuous at all points.

An important class of real-valued continuous functions of a single variable consists of those functions that are continuous on intervals. They have the following properties.

Weierstrass' first theorem: A function that is continuous on a closed interval is bounded on that interval.

Weierstrass' second theorem: A function that is continuous on a closed interval assumes on that interval a largest and a smallest value.

Cauchy's intermediate value theorem: A function that is continuous on a closed interval assumes on it any value between those at the end points.

The inverse function theorem: If a function is continuous and strictly monotone on an interval, then it has a single-valued inverse function,

Notes

which is also defined on an interval and is strictly monotone and continuous on it.

Cantor's theorem on uniform continuity: A function that is continuous on a closed interval is uniformly continuous on it.

Every function that is continuous on a closed interval can be uniformly approximated on it with arbitrary accuracy by an algebraic polynomial, and every function f that is continuous on $[0, 2\pi]$ and is such that $f(0) = f(2\pi)$ can be uniformly approximated on $[0, 2\pi]$ with arbitrary accuracy by trigonometric polynomials (see Weierstrass theorem on the approximation of functions).

The concept of a continuous function can be generalized to wider forms of functions, above all, to functions of several variables. The definition above is preserved formally if one understands by E a subset of an n-dimensional Euclidean space $\mathbf{R}^n$, by $|x - x_0|$ the distance between two points $x \in E$ and $x_0 \in E$, by $U(x_0, \delta)$ the δ-neighbourhood of x_0 in $\mathbf{R}^n$, and by

$$\lim_{n \to \infty} x_n = x_0$$

the limit of a sequence of points in $\mathbf{R}^n$. A function $f : E \to \mathbf{R}$, $E \subset \mathbf{R}$, of several variables $x_1, \ldots, x_n$ that is continuous at a point $x_0 = (x_1^{(0)}, \ldots, x_n^{(0)}) \in E$ is also called continuous at this point x_0 jointly in the variables $x_1, \ldots, x_n$, in contrast to functions of several variables that are continuous in the variables individually. A function $f : E \to \mathbf{R}$, $E \subset \mathbf{R}^n$, is called continuous at a point x_0 in, say, the variable x_1 if the restriction of f to the set

$$E \bigcap \left\{ x = (x_1, \ldots, x_n) : x_2 = x_2^{(0)}, \ldots, x_n = x_n^{(0)} \right\},$$

is continuous at $x_1^{(0)}$, that is, if the function $f(x_1, x_2^{(0)}, \ldots, x_n^{(0)})$ of the single variable x_1 is continuous at $x_1^{(0)}$. A function $f : E \to \mathbf{R}$, $E \subset \mathbf{R}^n$, $n \geq 2$, can be continuous at x in every variable $x_1, \ldots, x_n$, but need not be continuous at this point jointly in the variables.

The definition of a continuous function goes over directly to complex-valued functions. Only one has to interpret $|f(x) - f(x_0)|$ in the definition above as the norm of the complex number $f(x) - f(x_0)$ and

$$\lim_{n \to \infty} f(x_n) = f(x_0)$$

as the limit in the complex plane.

All these definitions are special cases of the more general concept of a continuous function f with as domain of definition a certain topological space X and with values in a certain topological space Y (see Continuous mapping).

Many properties of real-valued continuous functions of a single variable carry over to continuous mappings between topological spaces. A generalization of Weierstrass' theorem mentioned above: The continuous image of a compact topological space in a Hausdorff space is compact. A generalization of Cauchy's intermediate value theorem for a continuous function on a closed interval: A continuous image of a connected topological space in a topological space is also connected. A generalization of the theorem on the inverse function of a strictly monotone continuous function: A continuous one-to-one mapping of a compactum onto a Hausdorff space is a homeomorphism. A generalization of the theorem on the limit of a uniformly-convergent sequence of continuous functions: If $f_n : X \to Y$ is a uniformly-convergent sequence of mappings of a topological space X into a metric space Y that are continuous (at a point $x_0 \in X$) then the limit mapping $f = \lim_{n \to \infty} f_n$ is also continuous (at x_0). A generalization of Weierstrass' theorem on the approximation of functions that are continuous on a closed interval is the Stone–Weierstrass theorem.

Check in Progress-I

Q. 1 Define Continuous Mapping.

Solution

………………………………………………………………………..

…………………………………………………………………………

………………………………………………………………………….

…………………………………………………………………………

Notes

Q. 2 Give Example of Continuous mapping.

Solution

………………………………………………………………………..

…………………………………………………………………………

………………………………………………………………………….

…………………………………………………………………………

6.2 HOMEOMORPHISM

A one-to-one correspondence between two topological spaces such that the two mutually-inverse mappings defined by this correspondence are continuous. These mappings are said to be homeomorphic, or topological, mappings, and also homeomorphisms, while the spaces are said to belong to the same topological type or are said to be homeomorphic or topologically equivalent. They are isomorphic objects in the category of topological spaces and continuous mappings. A homeomorphism must not be confused with a condensation (a bijective continuous mapping); however, a condensation of a compactum onto a Hausdorff space is a homeomorphism.

Examples. 1) The function 1/(eX+1) establishes a homeomorphism between the real line RR and the interval (0,1); 2) a closed circle is homeomorphic to any closed convex polygon; 3) three-dimensional projective space is homeomorphic to the group of rotations of the space R3 around the origin and also to the space of unit tangent vectors to the sphere S2; 4) all compact zero-dimensional groups with a countable base are homeomorphic to the Cantor set; 5) all infinite-dimensional separable Banach spaces, and even all Fréchet spaces, are homeomorphic; 6) a sphere and a torus are not homeomorphic.

The term "homeomorphism" was introduced in 1895 by H. Poincaré [3], who applied it to (piecewise-) differentiable mappings of domains and submanifolds in RnRn; however, the concept was known earlier, e.g. to F. Klein (1872) and, in a rudimentary form, to A. Möbius (as an elementary likeness, 1863). At the beginning of the 20th century homeomorphisms began to be studied without assuming differentiability, as a result of the development of set theory and the axiomatic method.

This problem, which was explicitly stated for the first time by D. Hilbert [7], forms the content of Hilbert's fifth problem. Of special importance was the discovery by L.E.J. Brouwer that RnRn and RmRm are not homeomorphic if n≠mn≠m. This discovery restored the faith put by mathematicians in geometric intuition. This faith had been shaken by G. Cantor's result stating that Rn and Rm have the same cardinality and by the result obtained by G. Peano on the possibility of a continuous mapping from Rn onto Rm, n<m. The concepts of a metric (or, respectively, a topological) space, introduced by M. Fréchet and F. Hausdorff, laid a firm foundation for the concept of a homeomorphism and made it possible to formulate the concepts of a topological property (a property which remains unchanged under a homeomorphism), of topological invariance, etc., and to formulate the problem of classifying topological spaces of various types up to a homeomorphism. However, when presented in this manner, the problem becomes exceedingly complicated even for very narrow classes of spaces. In addition to the classical case of two-dimensional manifolds, such a classification was given only for certain types of graphs, for two-dimensional polyhedra and for certain classes of manifolds. The general problem of classification cannot be algorithmically solved at all, since it is impossible to obtain an algorithm for distinguishing, say, manifolds of dimension larger than three. Accordingly, the classification problem is usually posed in the framework of a weaker equivalence relation, e.g. in algebraic topology using homotopy type or, alternatively, to classify spaces having a certain specified structure. Even so, the homeomorphism problem remains highly important. In the topology of manifolds it was only in the late 1960s that methods for studying manifolds up to a homeomorphism were developed. These studies are carried out in close connection with homotopic, topological, piecewise-linear, and smooth structures.

A second problem is the topological characterization of individual spaces and classes of spaces (i.e. a specification of their characteristic topological properties, formulated in the language of general topology, cf. Topology, general and Topological invariant). This has been solved, for example, for one-dimensional manifolds, two-dimensional manifolds, Cantor sets, the Sierpiński curve, the Menger curve, pseudo-arcs, Baire spaces, etc. Spectra furnish a universal tool for the topological characterization of spaces; Aleksandrov's homeomorphism theorem was obtained using spectra [4]. The sphere and, in general, the class of locally Euclidean spaces, has been characterized by a sequence of subdivisions gradually diminishing in size [5]. A description of locally compact Hausdorff groups by means of spectra has been given [6]. Another method is to consider various algebraic structures connected with the mappings. Thus, a compact Hausdorff space is homeomorphic to the space of maximal ideals of the algebra of real functions defined on it. Many spaces are characterized by the semi-group of continuous mappings into themselves (cf. Homeomorphism group). In general topology a topological description is given of numerous classes of topological spaces. The characterization of spaces inside a given class is also of interest. Thus, it is very useful to describe a sphere as a compact manifold covered by two open cells. The problem of algorithmic identification of spaces has not been studied much. At the time of writing (1977) it has not been solved for the sphere SnSn where n≥3.

In general, the non-homeomorphism of two topological spaces is proved by specifying a topological property displayed by only one of them (compactness, connectedness, etc.; e.g., a segment differs from a circle in that it can be divided into two by one point); the method of invariants is especially significant in this connection. Invariants are either defined in an axiomatic manner for a whole class of spaces at the same time, or else algorithmically, according to a specific representation of the space, e.g. by triangulation, by the Heegaard diagram, by decomposition into handles (cf. Handle theory), etc. The problem in the former case is to compute the invariant, while in the latter it is to prove topological invariance. An intermediate case is also possible — e.g. characteristic classes (cf. Characteristic class) of smooth manifolds were at first defined as obstructions to the construction of vector and frame fields, and later as the image of the tangent bundle under mappings of the KOKO-functor into a cohomology functor, but in neither case can the respective problems be solved by definition. Historically the first example of proving topological invariance (of the linear dimension of RnRn) was given by Brouwer in 1912. The classical method, due to Poincaré, is to begin by giving both definitions — the "computable" and the "invariant" — and then to prove that they are identical. This method proved especially useful in the theory of homology of a polyhedron. Another method is to prove that an invariant remains unchanged under elementary transformations of a representation of the space (e.g. subdivision by triangulations). It is completed if it is known that it is possible to obtain all the representations of a given type in this manner. Thus, the so-called "Hauptvermutung" of combinatorial topology arose in the topology of polyhedra in this connection. This method (which was also proposed by Poincaré) proved highly useful in the topology of two and three dimensions, in particular in knot theory, but it is out of use now (except for the constructive direction) not so much because the "Hauptvermutung" proved to be untrue, as because the development of category theory made it possible to give more realistic definitions, more in accordance with the subject matter, with a more accurate presentation of the problem of computation and topological invariance. Thus, the invariance of homology groups, which are defined functorially for spaces but are defined in a computable manner for complexes, follows from the comparison of the category of complexes and homotopy classes of simplicial mappings with the category of homotopy classes of continuous mappings. In this way one does not have to give a separate definition for a large category and one can extend it to a smaller category as well. (The sources of this idea are found in Brouwer's theory of degree.) The superiority of the new method was seen to be particularly evident in connection with the second definition of characteristic classes, given above, as transformations of functors. Thus, the problem of topological invariance naturally turned out to be a part of the question of the relation between the KK-functor and its topological generalization.

If two spaces are homeomorphic, then the method of spectra (and of diminishing subdivisions) is the only one of general value for the establishment of homeomorphism. On the other hand, if a classification has already been constructed the problem is solved by comparison of invariants. In practice the establishment of homeomorphism often proves to be a very difficult geometrical problem, which must be solved by

Notes

employing special tools. Thus, homeomorphism of Euclidean spaces and some of their quotient spaces is established using a pseudo-isotopy.

6.2.1 Closed mapping

A mapping of one topological space to another, under which the image of every closed set is a closed set. The class of continuous closed mappings plays an important role in general topology and its applications. Continuous closed compact mappings are called perfect mappings. A continuous mapping $f: X \to Y$, $f(X) = Y$, of T_1-spaces is closed if and only if the decomposition $\{f^{-1}y: y \in Y\}$ is continuous in the sense of Aleksandrov (upper continuous) or if for every open set U in X, the set $f^{\#} = \{y \in Y: f^{-1}y \in U\}$ is open in U. The latter property is basic to the definition of upper semi-continuous many-valued mappings. That is, f is closed if and only if its (many-valued) inverse mapping is upper continuous. Any continuous mapping of a Hausdorff compactum onto a Hausdorff space is closed. Any continuous closed mapping of T_1-spaces is a quotient mapping; the converse is false. The orthogonal projection of a plane onto a straight line is continuous and open, but not closed. Similarly, not every continuous closed mapping is open. If $f: X \to Y$ is continuous and closed, with X and Y completely regular, then $\overline{f}\,\square^{-1}\, y = [f^{-1}y]\ \beta X$ for any point $y \in Y$. (Here βX is the Stone–Čech compactification and $\overline{f}: \beta X \to \beta Y$ is the continuous extension of the mapping to the Stone–Čech compactifications of X and Y); the converse is true in the class of normal spaces. Passage to the image under a continuous closed mapping preserves the following topological properties: normality; collection-wise normality; perfect normality; paracompactness; weak paracompactness. Complete regularity and strong paracompactness need not be preserved under continuous closed — and even perfect — mappings. Passage to the pre-image under a continuous closed mapping need not preserve the above-mentioned properties. The explanation for this is that the pre-image of a point under a continuous closed mapping need not be compact, though in many cases there is little difference between continuous closed and perfect mappings. If f is a continuous closed mapping of a metric space X onto a space Y satisfying the first axiom of countability, then Y is metrizable and the boundary of the pre-image $f^{-1}y$ is compact for every $y \in Y$. If f is a continuous closed mapping of a metric space X onto a T_1-space Y, then the set of all points $y \in Y$ for which $f^{-1}y$ is not compact is σ-discrete.

6.2.2 Open mapping

A mapping of one topological space into another under which the image of every open set is itself open.

Projections of topological products onto the factors are open mappings. Openness of a mapping can be interpreted as a form of continuity of its inverse many-valued mapping. A one-to-one continuous open mapping is a homeomorphism. In general topology, open mappings are used in the classification of spaces. The question of the behaviour of topological invariants under continuous open mappings is important. All spaces with the first axiom of countability, and only they, are images of metric spaces under continuous open mappings. A metrizable space which is the image of a complete metric space under a continuous open mapping is metrizable by a complete metric. If a paracompact space is the image of a complete metric space under a continuous open mapping, then it is metrizable. A countable-to-one continuous open mapping of compacta does not increase the dimensions. However, a 3-dimensional cube can be mapped by a continuous open mapping onto a cube of any larger dimension. Every compactum is the image of a certain one-dimensional compactum under a continuous open mapping with zero-dimensional fibres (i.e. inverse images of points)

Continuous open mappings under which the inverse images of all points are compact — the so-called compact-open mappings — are of separate interest in their own right. Spaces with a uniform base, and only they, are inverse images of metric spaces under compact-open mappings. Closed continuous open mappings are also important. All continuous open mappings of compacta into Hausdorff spaces (cf. Hausdorff space) fall into this category. Continuous closed open mappings preserve metrizability. Open mappings with discrete fibres play an important role in the theory of functions of one complex variable: these include all holomorphic functions in a domain. The theorem on the openness of holomorphic functions is central to proving the maximum-modulus principle, and to proving the fundamental theorem on the existence of a root of an arbitrary non-constant polynomial over the field of complex numbers.

6.2.3 Perfect mapping

A continuous closed mapping (cf. Closed mapping; Continuous mapping) of topological spaces under which the pre-image of every point is compact. Perfect mappings are akin to continuous mappings from compact spaces into Hausdorff spaces (every such mapping is perfect), although the scope of the definition covers all topological spaces. In the class of completely-regular spaces, the perfect mappings are characterized by the fact that their Stone–Čech extension maps remainders to remainders (cf. Remainder of a space). Perfect mappings preserve metrizability, paracompactness, weight, and Čech completeness. Other invariants (such as the character of a space) are transformed in a proper way. The class of perfect mappings is closed under taking products and composition. A restriction of a perfect mapping to a closed subspace is perfect (this is false for quotient and open mappings).

The above properties of perfect mappings have led to a situation where this class of mappings has begun to play a pivotal role in the classification of topological spaces. The completely-regular pre-images of metric spaces under perfect mappings are characterized as paracompact feathered p-) spaces (cf. Paracompact space; Feathered space). The class of paracompact pp-spaces is closed under perfect mappings and their inverses. An important property of perfect mappings is that they can be restricted to certain closed subspaces without reducing the image in such a way that the resulting mapping is irreducible, that is, it cannot be further restricted without reducing the image (cf. also Irreducible mapping). Irreducible perfect mappings are the starting point for constructing a theory of absolutes of topological spaces (cf. Absolute). For an irreducible perfect mapping, the ππ-weight (cf. Weight of a topological space) of the image is always equal to that of the pre-image, and the Suslin number of the image is equal to that of the pre-image. If a completely-regular T_1-space is mapped onto a completely-regular T_1-space by a perfect mapping, then X is homeomorphic to a closed subspace of the topological product of Y with some T_2-compactum. The diagonal product of a perfect mapping and a continuous mapping of T_2-spaces is always a perfect mapping; in particular, the diagonal product of a perfect mapping and a compression (i.e. a one-to-one continuous mapping onto) is a homeomorphism. If a topological space can be mapped perfectly onto one metric space and compressed onto another metric space (which need not be the same), then it is itself metrizable.

6.2.4 Quotient mapping

A mapping f of a topological space X onto a topological space Y for which a set $V \subseteq Y$ is open in Y if and only if its pre-image $f^{-1}(v)$ is open in X. If one is given a mapping f of a topological space X onto a set Y, then there is on Y a strongest topology T_f (that is, one containing the greatest number of open sets) among all the topologies relative to which f is continuous. The topology T_f consists of all sets $V \subseteq Y$ such that $f^{-1}(v)$ is open in X. This topology is the unique topology on Y such that f is a quotient mapping. Therefore T_f is called the quotient topology corresponding to the mapping f and the given topology T on X.

The construction described above arises in studying decompositions of topological spaces and leads to an important operation — passing from a given topological space to a new one — a decomposition space. Suppose one is given a decomposition γ of a topological space (X,T), that is, a family γ of non-empty pairwise-disjoint subsets of X that covers X. Then a projection mapping $\pi: X \to \gamma$ is defined by the rule: $\pi(x) = P \in \gamma$ if $x \in P \subseteq X$. The set γ is now endowed with the quotient topology T_π corresponding to the topology T on X and the mapping π, and (γ,T_π) is called a decomposition space of (X,T). Thus, up to a homeomorphism a circle can be represented as a decomposition space of a line segment, a sphere as a decomposition space of a disc, the Möbius band as a decomposition space of a rectangle, the projective plane as a decomposition space of a sphere, etc.

The following properties of quotient mappings, connected with considering diagrams, are important: Let $f:X\to Y$ be a continuous mapping with $f(X)=Y$. Then there are a topological space Z, a quotient mapping $g:X\to Z$ and a continuous one-to-one mapping (that is, a contraction) $h:Z\to Y$ such that $f=h\circ g$. For Z one can take the decomposition space $\gamma =\{ f^{-1}y:\ y\in Y\}$ of X into the complete pre-images of points under f, and the role of g is then played by the projection π. Suppose one is given a continuous mapping $f_2:X\to Y_2$ and a quotient mapping $f_1:X\to Y_1$, where the following condition is satisfied: If $x',x''\in X'$, and $f_1(x')=f_1(x'')$, then also $f_2(x')=f_2(x'')$. Then the unique mapping $g:Y_1\to Y_2$ such that $g\circ f_1=f_2$ turns out to be continuous. The restriction of a quotient mapping to a subspace need not be a quotient mapping — even if this subspace is both open and closed in the original space. The Cartesian product of a quotient mapping and the identity mapping need not be a quotient mapping, nor need the Cartesian square of a quotient mapping be such. The restriction of a quotient mapping to a complete pre-image does not have to be a quotient mapping. More precisely, if $f:X\to Y$ is a quotient mapping and if $Y_1\subseteq Y$, $X_1= f^{-1}(Y_1)$, $Y_1=f|_X$, then $f_1:X_1\to Y_1$ need not be a quotient mapping. This cannot occur if Y_1 is open or closed in Y.

These facts show that one must treat quotient mappings with care and that from the point of view of category theory the class of quotient mappings is not as harmonious and convenient as that of the continuous mappings, perfect mappings and open mappings (cf. Continuous mapping; Perfect mapping; Open mapping). However, the consideration of decomposition spaces and the "diagram" properties of quotient mappings mentioned above assure the class of quotient mappings of a position as one of the most important classes of mappings in topology.

This class contains all surjective, continuous, open or closed mappings (cf. Closed mapping). Quotient mappings play a vital role in the classification of spaces by the method of mappings. Thus, k-spaces are characterized as quotient spaces (that is, images under quotient mappings) of locally compact Hausdorff spaces, and sequential spaces are precisely the quotient spaces of metric spaces.

The majority of topological properties are not preserved under quotient mappings. Thus, a quotient space of a metric space need not be a Hausdorff space, and a quotient space of a separable metric space need not have a countable base. Therefore the question of the behaviour of topological properties under quotient mappings usually arises under additional restrictions on the pre-images of points or on the image space. It is known, for example, that if a compactum is homeomorphic to a decomposition space of a separable metric space, then the compactum is metrizable. Under a quotient mapping of a separable metric space on a regular T_1-space with the first axiom of countability, the image is metrizable. But there are topological invariants that are stable relative to any quotient mapping. These include, for example, sequentiality and an upper bound on tightness. In topological algebra quotient mappings that are at the same time algebra homeomorphisms often have much more structure than in general topology. Thus, an algebraic homomorphism of one topological group onto another that is a quotient mapping is necessarily an open mapping. Thanks to this, the range of topological properties preserved by quotient homomorphisms is rather broad (it includes, for example, metrizability).

6.3 STONE-WEIERSTRASS THEOREM

In mathematical analysis, the Weierstrass approximation theorem states that every continuous function defined on a closed interval $[a, b]$ can be uniformly approximated as closely as desired by a polynomial function. Because polynomials are among the simplest functions, and because computers can directly evaluate polynomials, this theorem has both practical and theoretical relevance, especially in polynomial interpolation. The original version of this result was established by Karl Weierstrass in 1885 using the Weierstrass transform.

Marshall H. Stone considerably generalized the theorem (Stone 1937) and simplified the proof (Stone 1948). His result is known as the Stone–Weierstrass theorem. The Stone–Weierstrass theorem generalizes the Weierstrass approximation theorem in two directions both regressive and progressive: instead of the real interval $[a, b]$, an arbitrary compact Hausdorff space X is considered, and instead of the algebra of polynomial functions, approximation with elements from more general subalgebras of $C(X)$ is investigated. The Stone–Weierstrass theorem is a vital result in the study of the algebra of continuous functions on a compact Hausdorff space.

Further, there is a generalization of the Stone–Weierstrass theorem to noncompact Tychonoff spaces, namely, any continuous function on a Tychonoff space is approximated uniformly on compact sets by algebras of the type appearing in the Stone–Weierstrass theorem and described below.

A different generalization of Weierstrass' original theorem is Mergelyan's theorem, which generalizes it to functions defined on certain subsets of the complex plane

Check In Progress-II

Q. 1 Define Quotient Mapping.

Solution

…………………………………………………………………………..

………………………………………………………………………………

……………………………………………………………………………….

………………………………………………………………………………

Q. 2 Give perfect mapping.

Solution

…………………………………………………………………………..

………………………………………………………………………………

……………………………………………………………………………….

………………………………………………………………………………

6.4 INFINITE PRODUCT THEOREM

Weierstrass' infinite product theorem [1]: For any given sequence of points in the complex plane $\mathbf{C}$,

$$0,\ldots,0,\ \alpha_1,\ \alpha_2,\ \ldots,$$

$$0 < |\alpha_k| \leq |\alpha_{k+1}|, \quad k = 1, 2, \ldots; \quad \lim_{k \to \infty} |\alpha_k| = \infty,$$

there exists an entire function with zeros at the points α_k of this sequence and only at these points. This function may be constructed as a canonical product:

$$W(z) = z^{\lambda} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right) e^{P_k(z)},$$

where λ is the multiplicity of zero in the sequence (1), and

$$P_k(z) = \frac{z}{\alpha_k} + \frac{z^2}{2\alpha_k^2} + \ldots + \frac{z^{m_k}}{2\alpha_k^{m_k}}.$$

The multipliers

$$W\left(\frac{z}{\alpha_k}; m_k\right) = \left(1 - \frac{z}{\alpha_k}\right) e^{P_k(z)}$$

are called Weierstrass prime multipliers or elementary factors. The exponents m_k are chosen so as to ensure the convergence of the product (2); for instance, the choice $m_k = k$ ensures the convergence of (2) for any sequence of the form (1).

It also follows from this theorem that any entire function $f(z)$ with zeros (1) has the form

$$f(z) = e^{g}(z) W(z),$$

where $W(z)$ is the canonical product (2) and $g(z)$ is an entire function (see also Hadamard theorem on entire functions).

Weierstrass' infinite product theorem can be generalized to the case of an arbitrary domain $D \subset \mathbf{C}$: Whatever a sequence of points $\{\alpha_k\} \subset D$ without limit points in D, there exists a holomorphic function f in D with zeros at the points α_k and only at these points.

The part of the theorem concerning the existence of an entire function with arbitrarily specified zeros may be generalized to functions of several complex variables as follows: Let each point α of the complex space $\mathbf{C}^n$, $n \geq 1$, be brought into correspondence with one of its neighbourhoods U_α and with a function f_α which is holomorphic in U_α. Moreover, suppose this is done in such a way that if the intersection $U_\alpha \cap U_\beta$ of the neighbourhoods of the points $\alpha, \beta \in \mathbf{C}^n$ is non-empty, then the fraction $f_\alpha / f_\beta \neq 0$ is a holomorphic function in $U_\alpha \cap U_\beta$. Under these conditions there exists an entire function f in $\mathbf{C}^n$ such that the fraction f / f_α is a holomorphic function at every point $\alpha \in \mathbf{C}^n$. This theorem is known as Cousin's second theorem (see also Cousin problems).

6.5 PREPARATION THEOREM

Weierstrass' preparation theorem. A theorem obtained and originally formulated by K. Weierstrass in 1860 as a preparation lemma, used in the proofs of the existence and analytic nature of the implicit function of a complex variable defined by an equation $f(z, w) = 0$ whose left-hand side is a holomorphic function of two complex variables. This theorem generalizes the following important property of holomorphic functions of one complex variable to functions of several complex variables: If $f(z)$ is a holomorphic function of z in a neighbourhood of the coordinate origin with $f(0) = 0$, $f(z) \not\equiv 0$, then it may be represented in the form $f(z) = z^s g(z)$, where s is the multiplicity of vanishing of $f(z)$ at the coordinate origin, $s \geq 1$, while the holomorphic function $g(z)$ is non-zero in a certain neighbourhood of the origin.

The formulation of the Weierstrass preparation theorem for functions of n complex variables, $n \geq 1$. Let

$$f(z) = f(z_1, ..., z_n)$$

be a holomorphic function of $z = (z_1, ..., z_n)$ in the polydisc

$$U = \{z : |z_i| < a_i, i = 1, ..., n\},$$

and let

$$f(0) = 0, \qquad f(0, ..., 0, z_n) \not\equiv 0 .$$

Then, in some polydisc

$$V = \{z : |z_i| < b_i \leq a_i, i = 1, ..., n\},$$

the function $f(z)$ can be represented in the form

$$f(z) = [z_n^s + f_1(z_1, ..., z_{n-1}) z_n^{s-1} + \dots$$
$$\dots + f_s(z_1, ..., z_{n-1})] g(z),$$

where s is the multiplicity of vanishing of the function

$$f(z_n) = f(0, ..., 0, z_n)$$

at the coordinate origin, $s \geq 1$; the functions $f_j(z_1, ..., z_{n-1})$ are holomorphic in the polydisc

$$V' = \{(z_1, ..., z_{n-1}) : |z_i| < b_i, i = 1, ..., n-1\},$$

$$f_j(0, ..., 0) = 0, \qquad j = 1, ..., s;$$

the function $g(z)$ is holomorphic and does not vanish in V . The functions $f_j(z_1, ..., z_{n-1})$, $j = 1, ..., s$, and $g(z)$ are uniquely determined by the conditions of the theorem.

If the formulation is suitably modified, the coordinate origin may be replaced by any point $a = (a_1, ..., a_n)$ of the complex space $\mathbf{C}^n$. It follows from the Weierstrass preparation theorem that for $n > 1$, as distinct from the case of one complex variable, every neighbourhood of a zero of a holomorphic function contains an infinite set of other zeros of this function.

Weierstrass' preparation theorem is purely algebraic, and may be formulated for formal power series. Let $\mathbf{C}[[z_1, ..., z_n]]$ be the ring of formal power series in the variables $z_1, ..., z_n$ with coefficients in the field of complex numbers $\mathbf{C}$; let f be a series of this ring whose terms

Notes

have lowest possible degree $s \geq 1$, and assume that a term of the form cz_n^s, $c \neq 0$, exists. The series f can then be represented as

$$f = (z_n^s + f_1 z_n^{s-1} + \ldots + f_s) g,$$

where $f_1, \ldots, f_s$ are series in $\mathbf{C}[[z_1, \ldots, z_{n-1}]]$ whose constant terms are zero, and g is a series in $\mathbf{C}[[z_1, \ldots, z_n]]$ with non-zero constant term. The formal power series $f_1, \ldots, f_s$ and g are uniquely determined by f.

A meaning which is sometimes given to the theorem is the following division theorem: Let the series

$$f \in \mathbf{C}[[z_1, \ldots, z_n]]$$

satisfy the conditions just specified, and let g be an arbitrary series in $\mathbf{C}[[z_1, \ldots, z_n]]$. Then there exists a series

$$h \in \mathbf{C}[[z_1, \ldots, z_n]]$$

and series

$$a_j \in \mathbf{C}[[z_1, \ldots, z_{n-1}]], \qquad a_j(0, \ldots, 0) = 0,$$

$$j = 0, \ldots, s-1,$$

which satisfy the following equation:

$$g = hf + a_0 + a_1 z_n + \ldots + a_{s-1} z_n^{s-1}.$$

Weierstrass' preparation theorem also applies to rings of formally bounded series. It provides a method of inductive transition, e.g. from $\mathbf{C}[[z_1, \ldots, z_{n-1}]]$ to $\mathbf{C}[[z_1, \ldots, z_n]]$. It is possible to establish certain properties of the rings $\mathbf{C}[z_1, \ldots, z_n]$ and $\mathbf{C}[[z_1, \ldots, z_n]]$ in this way, such as being Noetherian and having the unique factorization property. There exists a generalization of this theorem to differentiable functions [6]

6.6 SUMMARY

We study in this unit Weierstrass Preparation Theorem. We study infinite product theorem. We study continuous mapping theorem and its example. We study perfect mapping, quotient mapping and open mapping.

6.7 KEYWORD

MAPPING : An operation that associates each element of a given set (the domain) with one or more elements of a second set

UNIFORMIXABLE : adjective. causing fear, apprehension, or dread: a formidable opponent. of discouraging or awesome strength, size, difficulty, etc. ; intimidating: a formidable problem

6.8 EXERCISE

1. Prove that in a discrete metric space, every subset is both open and closed.

   If f is a map from a discrete metric space to any metric space, prove that f is continuous.

   Which maps from R (with its usual metric) to a discrete metric space are continuous ?
2. If f from R to R is a continuous map, is the image of an open set always open ?

   Is the inverse image of a closed set always closed ?
3. Show that in any metric space an ε-neighbourhood is an open set. Show that any open set can be written as a union of suitable ε-neighbourhoods.

   Give an example of an open subset of R (with its usual metric) which cannot be written as a union of finitely many ε-neighbourhoods.

   Can any open set can be written as a union of countably many suitable ε-neighbourhoods?
4. If f is a continuous function from R^2 to R (usual metrics!) prove that the set $\{ (x, y) \in R^2 \mid f(x, y) > 0 \}$ is an open subset of R^2. Deduce that the open unit disc and open unit square are open sets. Is the set $\{ (x, y) \in R^2 \mid f(x, y) \geq 0 \}$ necessarily a closed set ?

5. If (a_i) is a sequence in a metric space convergent to a point α, prove that α is the only limit point of the set $\{a_i\}$. Give an example of a set with exactly two limit points. Give an example of a set with countably many limit points.
6. Let X be the set {a, b, c, d, e}. Determine which of the following sets $\mathcal{T}$ are topologies on X.
   i. $\mathcal{T} = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$
   ii. $\mathcal{T} = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}\}$
   iii. $\mathcal{T} = \{X, \emptyset, \{a\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$
   iv. $\mathcal{T} = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$

6.9 ANSWER FOR CHECK IN PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 1

Q 2 Check in Section 1

Check in Progress-II

Answer Q. 1 Check in Section 4

Q 2 Check in Section 3

6.10 SUGGESTION READING AND REFERENCES

2. P.S. Aleksandrov, "Einführung in die Mengenlehre und die allgemeine Topologie" , Deutsch. Verlag Wissenschaft. (1984) (Translated from Russian)

3. N. Bourbaki, "Elements of mathematics. General topology" , Addison-Wesley (1966) (Translated from French)

4. J.L. Kelley, "General topology" , Springer (1975)

5. A.V. Arkhangel'skii, V.I. Ponomarev, "Fundamentals of

general topology: problems and exercises" , Reidel (1984) (Translated from Russian)

6. R.A. Aleksandryan, E.A. Mirzakhanyan, "General topology" , Moscow (1979) (In Russian)

7. L. Gillman, M. Jerison, "Rings of continuous functions" , v. Nostrand-Reinhold (1960)

8. P.S. Aleksandrov, "Einführung in die Mengenlehre und die allgemeine Topologie" , Deutsch. Verlag Wissenschaft. (1984) (Translated from Russian)

9. A.N. Kolmogorov, S.V. Fomin, "Elements of the theory of functions and functional analysis" , 1–2 , Graylock (1957–1961) (Translated from Russian)

10. S.M. Nikol'skii, "A course of mathematical analysis" , 1–2 , MIR (1977) (Translated from Russian)

11. V.A. Il'in, E.G. Poznyak, "Fundamentals of mathematical analysis" , 1–2 , MIR (1982) (Translated from Russian)

UNIT 7 TOPIC : TOPOLOGICAL MANIFOLDS

Structure

7.0 OBJECTIVE

In this topological manifold topic we learn induced homomorphism and indeed homomorphism. Learn patch work and Construction of chart. Learn clases of manifold and Gluing among boundary.

7.1 INTRODUCTION

The induced homomorphism is related to the study of the fundamental group. We will give a few theorems and notes, but first we make a definition.

Definition

Let X and Y be topological spaces; let x_0 be a point of X and let y_0 be a point of y. Suppose h is a continuous map from X to Y such that $h(x_0) = y_0$. Define a map h* from $\pi_1(X,x_0)$ to $\pi_2(Y,y_0)$ by composing a loop in

$\pi_1(X,x_0)$ with h to get a loop in $\pi_1(Y,y_0)$. Then h* is a homomorphism between fundamental groups known as the homomorphism induced by h.

Note 1

Let us check that if f is a loop in $\pi_1(X,x_0)$, then h*(f) is a loop in $\pi_1(Y,y_0)$. Note that h*(f) is a continuous map from [a,b] to Y (we will assume that [a,b] = [0,1] which has no difference to the general case since these two sets are homeomorphic), and h*(f(0)) = h*(x_0) = y0 and h*(f(1)) = h*(x_0) = y_0.

Note 2

We will check that h is indeed a homomorphism. To avoid repetition, whenever we call f and g loops, they will be known as loops based at x_0. Suppose f and g are two loops. Then [0 is the group operation on $\pi_1(X,x_0)$ and + is the group operation on $\pi_1(Y,y_0)$]

h*(f 0 g) = h*(f(2t)) for t in [0,1/2] = (h*(f)) + (h*(g))

h*(f 0 g) = h*(g(2t-1)) for t in [1/2,1] = (h*(f)) + (h*(g))

so that h* is indeed a homomorphism.

Note 3

Checking h* is a function (i.e, every loop in $\pi_1(X,x_0)$ gets mapped onto a unique loop in $\pi_1(Y,y_0)$) follows from the fact that if f and g are loops in π_1(X,x0) that are homotopic via the homotopy H, then h*(f) and h*(g) are homotopic via the homotopy h*H.

Note 4

Note that none of the above notes would be true unless h is continuous which is why this is needed in the hypothesis. We leave it to you to work out why h(x_0) = y_0 [which is fairly trivial].

We now prove one important theorem which can be used to check whether two topological spaces are homeomorphic or not. In fact, this theorem illustrates why algebraic topology was invented in the first place.

7.1.1 Applications of the Theorems

Theorem

Suppose X and Y are two homeomorphic topological spaces. If h is a homeomorphism from X to Y, then the induced homomorphism, h* is an isomorphism between fundamental groups [Assume that we are considering the fundamental groups $\pi_1(X,x_0)$ and $\pi_1(Y,y_0)$ with $h(x_0) = y_0$]

Proof

We have already checked in note 2 that h* is a homomorphism. It remains to check that h* is bijective. Suppose p is the inverse of h; then p* is the inverse of h*. This follows from the fact that (p(h))*(f) = p*(h*(f)) = f = (h(p))*(f) = h*(p*(f)). If f and g are two loops in X where f is not homotopic to g, the h*(f) is not homotopic to h*(g); if F is a homotopy between them, p*(F) would be a homotopy between f and g. If k is any loop in $\pi_1(Y,y_0)$, then h*(p*(k)) = k where p*(k) is a loop in X. This shows that h* is bijective.

Note: It is a good exercise to check that we used all the properties that h satisfies, i.e we used completely, the fact that h is a homeomorphism.

1. The torus is not homeomorphic to R^2 for their fundamental groups are not isomorphic (their fundamental groups don't have the same cardinality). Note that, a simply connected space cannot be homeomorphic to a non-simply connected space; one has a trivial fundamental group and the other does not.

2. In fact, any two topological spaces have homomorphic fundamental groups (at a particular base point). See note 2 where we may let h* be the homomorphism induced by the constant map. However, they need not have isomorphic fundamental groups (at a particular base point). This is interesting because it shows that the fundamental groups of any two topological spaces always have the same 'group structure'.

3. The fundamental group of the unit circle is isomorphic to the additive group of integers. Therefore, the fundamental group of [0,1] is isomorphic to the set of integers since [0,1] and the unit circle are homeomorphic (why is this statement false?). The one-point compactification of R also has a fundamental group isomorphic to the set of integers (since the one-point compactification of R is homeomorphic to the unit circle).

4. The converse of the theorem need not hold. For example, R^2 and R^3 have isomorphic fundamental groups but are still not homeomorphic. Their fundamental groups are isomorphic because each space is simply connected. However, the two spaces cannot be homeomorphic because deleting a point from R^2 leaves a non-simply connected space but deleting a point from R^3 leaves a simply connected space (If we delete a line lying in R^3, the space wouldn't be simply connected anymore. In fact this generalizes to R^n whereby deleting a (n-2) dimensional parallelopiped from R^n leaves a non-simply connected space).

Example 1

Show that the topological spaces $(0,1)$ **and** $(0,\infty)$ **(with their topologies being the unions of open balls resulting from the usual Euclidean metric on these subsets of** R**) are homeomorphic.**

To show that these two topological spaces are homeomorphic we must find a continuous bijection $f:X\to Y$ such that $f-1$ is also continuous.

Consider the following function $f:(0,1)\to(1,\infty)$ given by:

(1)

$$f(x)=1/x$$

We first show that f is bijection. Let $x,y\in(0,1)$ and suppose that $f(x)=f(y)$. Then:

(2)

$$1/x=1/y$$

Cross multiplying gives us that then $x=y$, so f is injective.

Now let $b\in(1,\infty)$. Since $b>1$ we have that $0<1b<1$, and so let $a=1b$. Then:

(3)

$$f(a)=1/a=1/1/b=b$$

So for all $b\in(1,\infty)$ there exists an $a\in(0,1)$ such that $f(a)=b$, so f is surjective.

It's not hard to see that f is a continuous map.

Furthermore, $f-1:(1,\infty)\to(0,1)$ is also given by $f-1(x)=1x$ (which is continuous), and so f is a homeomorphism between (0,1) and $(1,\infty)$, so these spaces are homeomorphic.

Example 2

Show that the spaces $(-r,r)$, $r>0$ and R with the topologies obtained by the unions of open balls with respect to the usual Euclidean metric are homeomorphic.

Consider the following function $f:(-r,r)\to$R given by:

(4)

$$f(x)=\tan(\pi x2r)$$

Then f is clearly continuous as f will always have the following form:

Further it should be clear that $f-1$ will always always be continuous:

Therefore f is a homeomorphism between $(-r,r)$ and R so these spaces are homeomorphic

7.2 HOMEOMORPHISM

A function f:X→Y is said to be a homeomorphism (topological mapping) if and only if the following conditions are satisfied:

(1) f is bijective

(2) f is continuous

(3) f^{-1} is continuous

It may be noted that if f is a homeomorphism from X to Y, then X is said to be homeomorphic to Y and is denoted by X≃Y. From the definition of a homeomorphism, it follows that X and Y are homeomorphic spaces, then their points and open sets are put into one-to-one correspondence. In other words, X and Ydiffer only in the nature of their points, but from the point of view of the subject of topology they are identical or have the same topological structure.

Remarks: "Homeomorphism" helps reduce complicated problems into simple form, that is, an apparently complicated space may possibly be homeomorphic to some space more familiar to us. Hence in this way, one determines the properties of complicated spaces easily.

Theorems

- Bijective continuous mapping $f:X\to Y$ is open if and only if f^{-1} is continuous.
- If X and Y are topological spaces, let $X\simeq Y$ mean that X and Y are homeomorphic. Then this relation is reflexive, symmetric and transitive.
- Let X and Y be topological spaces and $f:X\to Y$ be a bijective function, then the following are equivalent: (1) f is a homeomorphism; (2) for any subset U of X, f(U) is open in Y if and only if U is open in X; (3) for any subset C of X, f(C) is closed in Y if and only if C is closed in X; (4) for any subset A of X, $f(\overline{A})=\overline{f(A)}$

7.2.1 Topological Property

A property P is said to be a topological property if whenever a space X has the property P, all spaces which are homeomorphic to X also have the property P, $X\simeq Y\simeq Z$.

In other words, a topological property is a property which, if possessed by a topological space, is also possessed by all topological spaces homeomorphic to that space.

Note: It may noted that length, angle, boundedness, Cauchy sequence, straightness and being triangular or circular are not topological properties, whereas limit point, interior, neighborhood, boundary, first and second countability, and separablility are topological properties. We shall come across several topological properties in a following post. Because of its critical role the subject topology, it is usually described as the study of topological properties.

Examples:

• Let X=]−1, and f:X→R be defined by f(x)=tan(πx2). Then f is a homeomorphism and therefore]−1,1[≃R]−1,1[≃R. Note that]−1,1[]−1,1[and R have different lengths, therefore length is not a topological property. Also X is bounded and R is not bounded, therefore boundeness is not a topological property.

• Let f:]0,∞[→]0,∞[defined by f(x)=1xf(x)=1x, then f is a homeomorphism. Consider the sequences (xn)=(1,12,13,⋯) and (f(xn))=(1,2,3,…) in]0,∞[. (xn) is a Cauchy sequence, where (f(xn)) is not. Therefore, being a Cauchy sequence is not a topological property.

• Straightness is not a topological property, for a line may be bent and stretched until it is wiggly.

• Being triangular is not a topological property since a triangle can be continuously deformed into a circle and conversely.

7.2.2 Product Topology

Products of Sets

If X1 and X2 are two non-empty sets, then the Cartesian product X1×X2 is defined as X1×X2={(xi,xj:xi∈X1,xj∈X2)}

Projection Maps

Let A and B be non-empty sets, then they can be defined by the following two functions:

(1) p1:A×B→A. defined as p1(a,b)=ap1(a,b)=a for all (a,b)∈A×B.

(2) p2:A×B→B defined as p2(a,b)=bp2(a,b)=b for all (a,b)∈A×B(a,b)∈A×B.

The above maps are called the projection maps on A and B respectively.

Note: Let X1,X2,X3,…Xn be non-empty sets, then the projection maps p1,p2,p3,…pn be defined similarly.

Product Topology

Let X1×X2be the product of topological spaces X1 and X2. The coarsest topology τ on X1×X2 with respect to which the projection maps p1:X1×X2→X1 and p2:X1×X2→X2 are continuous, is said to be a product topology and thus the space (X1×X2,τ) is said to be the product space.

Remarks

7.2.3 Quotient Topology

• It may be observed that if X1 and X2 are distinct topological spaces then the collection S={p_1−(G1)$_\lambda{}^x$p−1(G2):G1∈τ1,G2∈τ2} form a subbase for product topology on X1×X2.
• It may be noted that if A and B are any open interval, then A×B will be open rectangle strips. A collection of open rectangles form a basis for the usual topology on R2. So, generalizing this fact to the product of a finite number of spaces (X1,τ1) and (X2,τ2) are topological spaces then B={G1×G2:G1∈τ1,G2∈τ2} form a basis for product topology.

In this section, we will introduce a new way of constructing topological spaces called the *quotient construction*. This is intended to formalise pictures like the familiar picture of the 2-torus as a square with its opposite sides identified.

Mathematically, the square is a subset of the plane and specifying the identification of opposite sides means giving an *equivalence relation* on this set.

Given a space X and an equivalence relation ∼ on X, the quotient set X/∼ (the set of equivalence classes) inherits a topology called the *quotient topology*. Let q:X→X/∼ be the *quotient map* sending a point x to its equivalence class [x]; the quotient topology is defined to be the *most refined topology* on X/∼ (i.e. the one with the largest number of open sets) for which q is continuous.

If you try to add too many open sets to the quotient topology, their preimages under q may fail to be open, so the quotient map will fail to be continuous. More concretely, a subset U⊂X/∼ is open in the quotient topology *if and only if* q−1(U)⊂X is open.

There is a ``most refined topology for which q is continuous", equivalently the collection of subsets U for which q−1(U) is open forms a topology.

The empty set is open: its preimage is the empty set which is open in X.

The whole space X/∼ is open: its preimage is the whole space X, which is open in X.

The preimage of a union is the union of the preimages, so if U is a collection of open sets in X/∼X/∼ then ⋃U∈U,U is open because its preimage under q is ⋃U∈Uq−1(U) which is a union of open sets in X.

The preimage of an intersection is the intersection of the preimages, so similarly the quotient topology allows finite intersections of open sets.

Examples

I want to give you some examples which indicate why this definition captures the intuitive idea we have of forming a topological space by making identifications.

Let X=[0,1] and let ∼ be the equivalence relation which has as equivalence classes singleton sets {x} for x∈(0,1) and the set {0,1}. The quotient space should be the circle, where we have identified the endpoints of the interval. Indeed, we can map X to the unit circle S1⊂C via the map q(x)=e2πix: this map takes 0 and 1 to 1∈S1 and is bijective elsewhere, so it is true that S1 is the set-theoretic quotient. We want to see that the usual topology on the circle is the quotient topology. Here are some open sets in S1 and their preimages under qq:

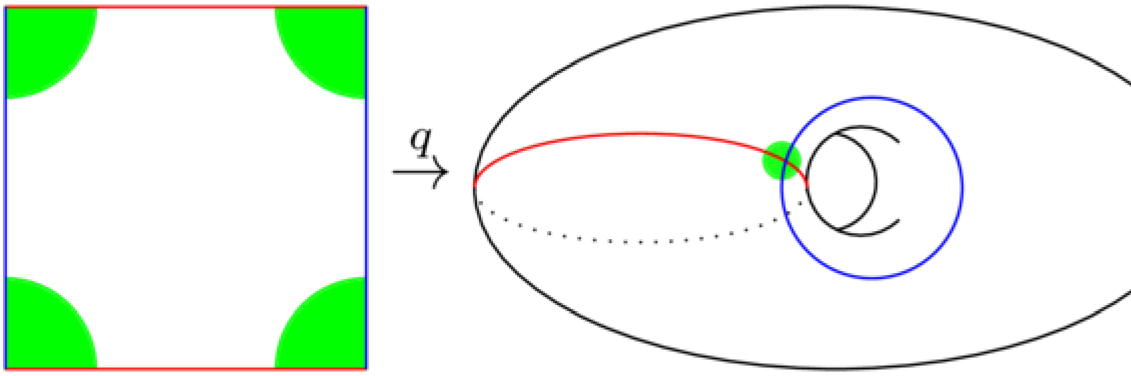
- If I take an arc in S1 which does not pass through the point 1∈S then its preimage is an interval in (0,1). The arc in the circle (in the subspace topology coming from C is open if and only if it is obtained by intersecting the circle with an open ball; in particular, it doesn't contain its endpoints. Similarly, the interval in (0,1) is open if and only if it doesn't contain its endpoints, so at least for these arcs in S1∖{1} we see that the arc is open if and only if its q-preimage is open.
- If I take an arc in S1 which does pass through 1∈S1 then its preimage is a pair of intervals [0,a}∪{b,1], which is open *in the subspace topology on* [0,1]⊂R if and only if the brackets {,} are

open brackets (the fact that the endpoints 0 and 1 are included does not stop it from being an open set: remember that [0,1] is an open set in [0,1]). These brackets are open if and only if the arc in S1 does not contain its endpoints, if and only if it is open in the subspace topology on S1.

The quotient space is therefore making formal the notion that *when you walk off the end*

of the interval, you come back at the other end: the preimage of an open set which crosses the point 1 is a pair of open sets at either end of the interval.

Let S=[0,1]×[0,1] be the square in R2 and let ~ be the equivalence relation which identifies opposite sides (red with red, blue with blue in the figure below). We know this is supposed to be the 2-torus:



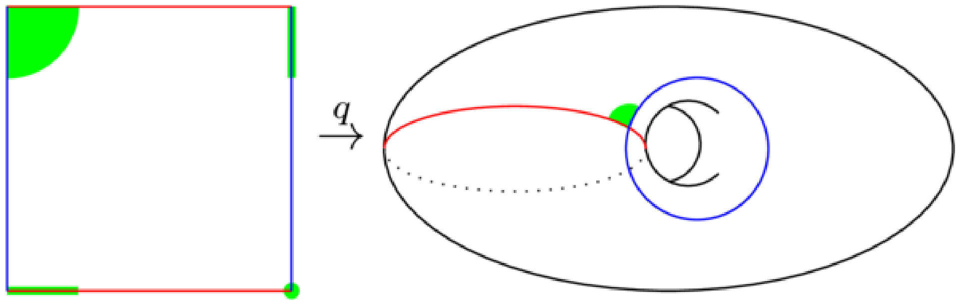
What is the preimage of the open green disc in the torus? It is a collection of four green quarter-discs at the corners of the square. This is open in the subspace topology on the square.

What happens if I take one of these quarter discs and look at its image in the torus? Clearly from looking at the picture, this should *not* give me an open set: the quarter disc includes the two closed intervals where it intersects the boundary of the square, which is fine when considered in the subspace topology on the square, but we don't want an open subset of the torus to look like a quarter-disc which is open along its curved edge and closed on its two straight edges.

The quotient topology fixes this for us. The preimage of this quarter-disc Vin the torus certainly contains the quarter-disc in the square. But it also contains:

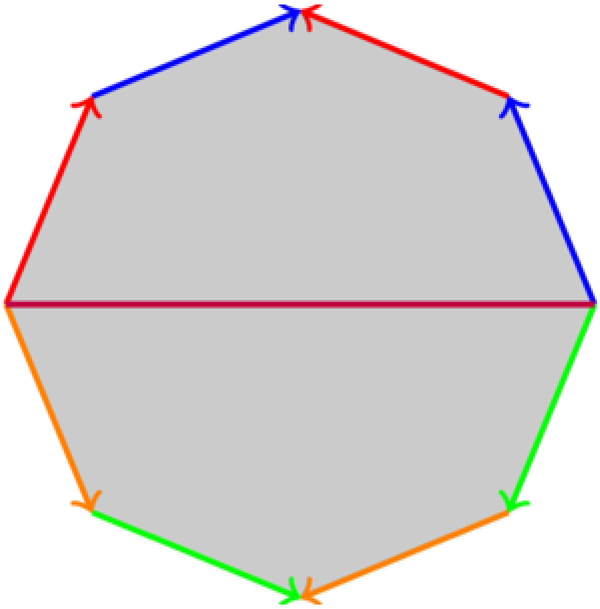
- two further intervals which are identified with sides of the quarter-disc under the equivalence relation,
- the remaining vertex of the square, which is identified with the centre of the quarter-disc under the equivalence relation.

That is definitely not an open set in the square, so q−1(V)) is not open,

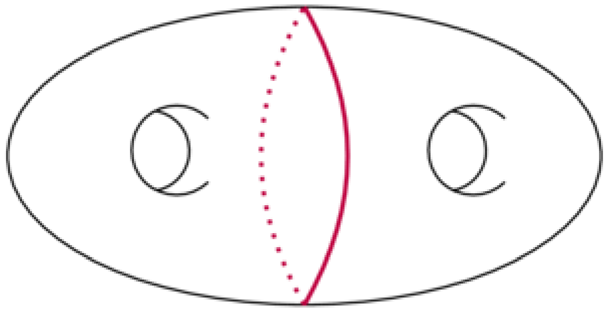


so V is not open.

Take an octagon with side identifications as in the figure (ignore the horizontal purple line for now):



The quotient space turns out to be a surface of genus 2:



You can understand this as follows: the purple line in the diagram becomes a circle in the quotient which slices the surface into two punctured tori; each half of the octagon is (topologically) a square (with the side identifications for the torus) with a puncture.

More generally, you can take a quotient of a $4g$-gon to get a genus g surface.

7.3 CRUSHING A SUBSPACE TO A POINT

) Let X be a space and A⊂X be a subspace. Let ∼∼ be the equivalence relation whose equivalence classes are A and {x}for x∉Ax. The quotient space X/∼ is usually written X/A: we think of this as the space obtained from X by crushing A down to a single point.

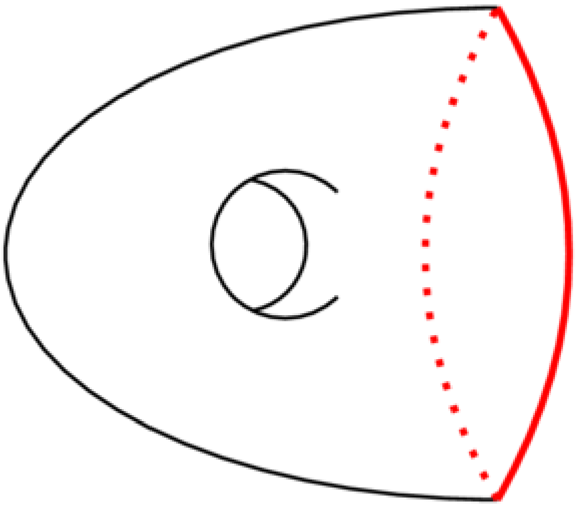
If X=[0,1] and A={0,1} then X/A=S1.

If X=D2 is the 2-disc and A=∂D2 (the boundary circle) then X/A=S2 (if we think of the centre of the disc as the North Pole then all the points in A are identified to get the South Pole.

Let X=S1×S1 be the 2-torus and let A=S1×{pt} be a meridian circle on the torus (the red circle in the figure). The quotient X/A is a *pinched torus*: see the figure below.

7.3.1 Pre-class questions

1. Let X be the space in the figure below and let A be the red subset. What is the topological space X/A?



Check In Progress-I

Q. 1 Define Product Topology.

Solution :

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

……………………………………………………………………………

Q. 2 Define Homomophism

Solution :

……………………………………………………………………………

……………………………………………………………………………

7.4 TOPOLOGICAL MANIFOLD

In topology, a branch of mathematics, a **topological manifold** is a topological space (which may also be a separated space) which locally resembles real *n*-dimensional space in a sense defined below. Topological manifolds form an important class of topological spaces with applications throughout mathematics. All manifolds are topological manifolds by definition, but many manifolds may be equipped with additional structure (e.g. differentiable manifolds are topological manifolds equipped with a differential structure). Every manifold has an "underlying" topological manifold, gotten by simply "forgetting" any additional structure the manifold has.

7.4.1 Manifold

On a sphere, the sum of the angles of a triangle is not equal to 180°. A sphere is not a Euclidean space, but locally the laws of the Euclidean geometry are good approximations. In a small triangle on the face of the earth, the sum of the angles is very nearly 180°. A sphere can be represented by a collection of two dimensional maps, therefore a sphere is a manifold.

A manifold is an abstract mathematical space in which every point has a neighbourhood which resembles Euclidean space, but in which the global structure may be more complicated. In discussing manifolds, the idea of dimension is important. For example, lines are one-dimensional, and planes are two-dimensional.

In a one-dimensional manifold (or one-manifold), every point has a neighbourhood that looks like a segment of a line. Examples of one-manifolds include a line, a circle, and two separate circles. In a two-manifold, every point has a neighbourhood that looks like a disk. Examples include a plane, the surface of a sphere, and the surface of a torus.

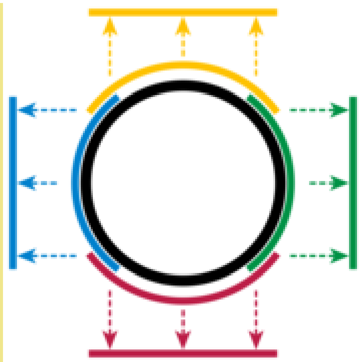
Manifolds are important objects in mathematics and physics because they allow more complicated structures to be expressed and understood in terms of the relatively well-understood properties of simpler spaces.

Additional structures are often defined on manifolds. Examples of manifolds with additional structure include differentiable manifolds on which one can do calculus, Riemannian manifolds on which distances and angles can be defined, symplectic manifolds which serve as the phase space in classical mechanics, and the four-dimensional pseudo-Riemannian manifolds which model space-time in general relativity.

A technical mathematical definition of a manifold is given below. To fully understand the mathematics behind manifolds, it is necessary to know elementary concepts regarding sets and functions, and helpful to have a working knowledge of calculus and topology.

7.4.2 Motivational examples

7.4.2.1 Circle



The four charts each map part of the circle to an open interval, and together cover the whole circle. The origin is understood to be at the centre of the circle.

The circle is the simplest example of a topological manifold after a line. Topology ignores bending, so a small piece of a circle is exactly the same as a small piece of a line. Consider, for instance, the top half of the unit circle, $x^2 + y^2 = 1$, where the y-coordinate is positive (indicated by the yellow arc in *Figure 1*). Any point of this semicircle can be uniquely described by its x-coordinate. So, by projecting onto the first coordinate, one obtains a continuous mapping between the semicircle and the open interval $(-1, 1)$:

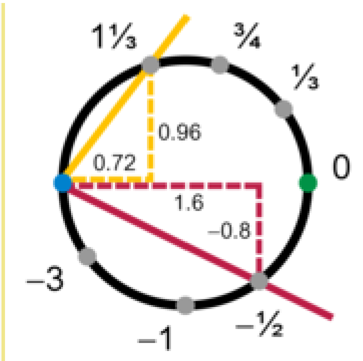
$$\chi_{\text{top}}(x, y) = x.$$

Such a function is called a *chart*. Similarly, there are charts for the bottom (red), left (blue), and right (green) parts of the circle. Together, these parts cover the whole circle and the four charts form an atlas for the circle.

The top and right charts overlap: their intersection lies in the quarter of the circle where both the *x*- and the *y*-coordinates are positive. The two charts χ_{top} and χ_{right} each map this part bijectively to the interval (0, 1). Thus a function *T* from (0, 1) to itself can be constructed, which first inverts the yellow chart to reach the circle and then follows the green chart back to the interval. Let *a* be any number in (0, 1), then:

$$T(a) = \chi_{\text{right}}\left(\chi_{\text{top}}^{-1}(a)\right) = \chi_{\text{right}}\left(a, \sqrt{1-a^2}\right) = \sqrt{1-a^2}.$$

Such a function is called a *transition map*.



A circle manifold chart based on slope, covering all but one point of the circle.

The top, bottom, left, and right charts show that the circle is a manifold, but they do not form the only possible atlas. Charts need not be geometric projections, and the number of charts is a matter of some choice. Consider the charts

$$\chi_{\text{minus}}(x, y) = s = \frac{y}{1+x}$$

and

$$\chi_{\text{plus}}(x, y) = t = \frac{y}{1-x}.$$

Here s is the slope of the line through the point at coordinates (x,y) and the fixed pivot point (−1,0); t is the mirror image, with pivot point (+1,0). The inverse mapping from s to (x,y) is given by

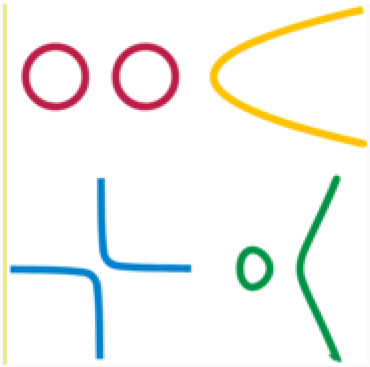
$$x = \frac{1-s^2}{1+s^2}, \qquad y = \frac{2s}{1+s^2};$$

it can easily be confirmed that $x^2+y^2 = 1$ for all values of the slope s. These two charts provide a second atlas for the circle, with

$$t = \frac{1}{s}.$$

Each chart omits a single point, either (−1,0) for s or (+1,0) for t, so neither chart alone is sufficient to cover the whole circle. It is not possible to cover the full circle with a single chart, since the circle is doubly connected and the line is only simply connected. Note that it is possible to construct a circle by "gluing" together a single piece of the line; this does not produce a chart, since a portion of the circle will be mapped to both "glued" regions at once.

7.4.2.2 Other curve



Four manifolds from algebraic curves: ■ circles, ■ parabola, ■ hyperbola, ■ cubic.

Manifolds need not be connected (all in "one piece"); thus a pair of separate circles is also a manifold. They need not be closed; thus a line segment without its ends is a manifold. And they need not be finite; thus a parabola is a manifold. Putting these freedoms together, two other example manifolds are a hyperbola (two open, infinite pieces) and the locus of points on the cubic curve $y^2 = x^3 - x$ (a closed loop piece and an open, infinite piece).

However, we exclude examples like two touching circles that share a point to form a figure-8; at the shared point we cannot create a satisfactory chart. Even with the bending allowed by topology, the vicinity of the shared point looks like a "+", not a line.

7.4.2.3 Enriched circle

Viewed using calculus, the circle transition function T is simply a function between open intervals, which gives a meaning to the statement that T is differentiable. The transition map T, and all the others, are differentiable on (0, 1); therefore, with this atlas the circle is a *differentiable manifold*. It is also *smooth* and *analytic* because the transition functions have these properties as well.

Other circle properties allow it to meet the requirements of more specialized types of manifold. For example, the circle has a notion of distance between two points, the arc-length between the points; hence it is a *Riemannian manifold*.

The study of manifolds combines many important areas of mathematics: it generalizes concepts such as curves and surfaces as well as ideas from linear algebra and topology. Certain special classes of manifolds also have additional algebraic structure; they may behave like groups, for instance.

Before the modern concept of a manifold there were several important results.

Carl Friedrich Gauss may have been the first to consider abstract spaces as mathematical objects in their own right. His theorema egregium gives a method for computing the curvature of a surface without considering the ambient space in which the surface lies. Such a surface would, in modern terminology, be called a manifold; and in modern terms, the theorem proved that the curvature of the surface is an intrinsic property. Manifold theory has come to focus exclusively on these intrinsic properties (or invariants), while largely ignoring the extrinsic properties of the ambient space.

Another, more topological example of an intrinsic property of a manifold is the Euler characteristic. For a non-intersecting graph in Euclidean 2-dimensional space, with *V* vertices (or corners), *E* edges and *F* faces (counting the exterior) Euler showed that $V-E+F = 2$. Thus 2 is called the Euler characteristic of Euclidean 2-dimensional space. By contrast, the Euler characteristic of the torus is 0, since the complete graph on seven points can be embedded into the torus. The Euler characteristic of other 2-dimensional spaces is a useful topological invariant, which can be extended to higher dimensions using Betti numbers.

Non-Euclidean geometry considers spaces where Euclid's parallel postulate fails. Saccheri first studied them in 1733. Lobachevsky, Bolyai, and Riemann developed them 100 years later. Their research uncovered two types of spaces whose geometric structures differ from that of classical Euclidean space; these gave rise to hyperbolic geometry and elliptic geometry. In the modern theory of manifolds, these notions correspond to manifolds with constant negative and positive curvature, respectively.

7.4.2.4 Synthesis

Bernhard Riemann was the first to do extensive work generalizing the idea of a surface to higher dimensions. The name *manifold* comes from Riemann's original German term, *Mannigfaltigkeit*, which William Kingdon Clifford translated as "manifoldness". In his Göttingen inaugural lecture, Riemann described the set of all possible values of a variable with certain constraints as a *Mannigfaltigkeit*, because the variable can have *many* values. He distinguishes between *stetige Mannigfaltigkeit* and *diskrete Mannigfaltigkeit* (*continuous manifoldness* and *discontinuous manifoldness*), depending on whether the value changes continuously or not. As continuous examples, Riemann refers to not only colors and the locations of objects in space, but also the possible shapes of a spatial figure. Using induction, Riemann constructs an *n-fach ausgedehnte Mannigfaltigkeit* (*n times extended manifoldness* or *n-dimensional manifoldness*) as a continuous stack of (n−1) dimensional manifoldnesses. Riemann's intuitive notion of a *Mannigfaltigkeit* evolved into what is today formalized as a manifold. Riemannian manifolds and Riemann surfaces are named after Bernhard Riemann.

In the study of complex variables, the process of analytic continuation leads to the construction of manifolds.

Abelian varieties were already implicitly known in Riemann's time as complex manifolds. Lagrangian mechanics and Hamiltonian mechanics, when considered geometrically, are also naturally manifold theories. All these use the notion of several characteristic axes or dimensions (known as generalized coordinates in the latter two cases), but these dimensions do not lie along the physical dimensions of width, height, and breadth.

Henri Poincaré studied three-dimensional manifolds and raised a question, today known as the Poincaré conjecture. As of 2006, a consensus among experts is that recent work by Grigori Perelman may have answered this question, after nearly a century of effort by many mathematicians.

Hermann Weyl gave an intrinsic definition for differentiable manifolds in 1912. During the 1930s Hassler Whitney and others clarified the foundational aspects of the subject, and thus intuitions dating back to the latter half of the 19th century became precise, and developed through differential geometry and Lie group theory.

Mathematical definition

In topology, an ***n*-manifold** is a second countable Hausdorff space in which every point has a neighbourhood homeomorphic to an open Euclidean *n*-ball,

$$\mathbf{B}^n = \{(x_1, x_2, ..., x_n) | x_1^2 + x_2^2 + ... + x_n^2 < 1\}.$$

Unless otherwise stated, a manifold is an *n*-manifold for some positive integer *n*, perhaps with additional structure. However, some authors admit manifolds which are not *n*-manifolds, in the sense that they allow different connected components to have different topological dimension.

The second countable condition excludes spaces such as the long line. The Hausdorff condition avoids spaces such as the one formed by identifying two real lines at every point except the origin:

$\mathbf{R} \times \{a\}$ and $\mathbf{R} \times \{b\}$

with the equivalence relation

$$(x,a) \sim (x,b) \text{ if } x \neq 0$$

which has a single point for each nonzero real number *r* plus two points 0_a and 0_b. In this space all neighbourhoods of 0_a intersect all neighbourhoods of 0_b.

There are many different kinds of manifold. All manifolds are topological manifolds, which locally have the topology of some Euclidean space. If there is additional structure, the structure on each map must be consistent with the overlapping maps. Differentiable manifolds have homeomorphisms on overlapping neighborhoods diffeomorphic with each other, so that the manifold has a well-defined set of functions which are differentiable in each neighbourhood, and so differentiable on the manifold as a whole.

7.5 CHARTS, ATLASES, AND TRANSITION MAPS

The spherical Earth is navigated using flat maps or charts, collected in an atlas. Similarly, a differentiable manifold can be described using mathematical maps, called *coordinate charts*, collected in a mathematical *atlas*. It is not generally possible to describe a manifold with just one chart, because the global structure of the manifold is different from the simple structure of the charts. For example, no single flat map can properly represent the entire Earth. When a manifold is constructed from multiple overlapping charts, the regions where they overlap carry information essential to understanding the global structure.

7.5.1 Charts

A **coordinate map**, a **coordinate chart**, or simply a **chart**, of a manifold is an invertible map between a subset of the manifold and a simple space such that both the map and its inverse preserve the desired structure. For a topological manifold, the simple space is some Euclidean space $\mathbf{R}^n$ and interest focuses on the topological structure. This structure is preserved by homeomorphisms, invertible maps that are continuous in both directions.

In the case of a differentiable manifold, a set of **charts** called an **atlas** allows us to do calculus on manifolds. Polar coordinates, for example, form a chart for the plane $\mathbf{R}^2$ minus the positive *x*-axis and the origin. Another example of a chart is the map χ_{top} mentioned in the section above, a chart for the circle.

7.5.2 Atlases

The description of most manifolds requires more than one chart (a single chart is adequate for only the simplest manifolds). A specific collection of charts which covers a manifold is called an **atlas**. An atlas is not unique as all manifolds can be covered multiple ways using different combinations of charts.

The atlas containing all possible charts consistent with a given atlas is called the **maximal atlas**. Unlike an ordinary atlas, the maximal atlas of a given atlas is unique. Though it is useful for definitions, it is a very abstract object and not used directly (e.g. in calculations).

7.5.3 Transition Maps

Charts in an atlas may overlap and a single point of a manifold may be represented in several charts. If two charts overlap, parts of them represent the same region of the manifold, just as a map of Europe and a map of Asia may both contain Moscow. Given two overlapping charts, a transition function can be defined which goes from an open ball in R^n to the manifold and then back to another (or perhaps the same) open ball in R^n. The resultant map, like the map *T* in the circle example above, is called a change of coordinates, a coordinate transformation, a transition function, or a transition map.

7.5.4 Additional Structure

An atlas can also be used to define additional structure on the manifold. The structure is first defined on each chart separately. If all the transition maps are compatible with this structure, the structure transfers to the manifold.

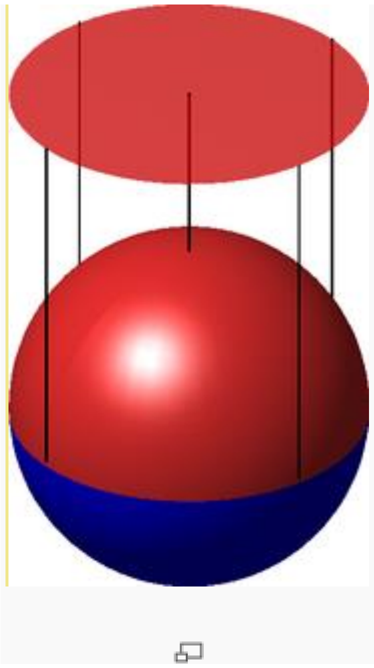
This is the standard way differentiable manifolds are defined. If the transition functions of an atlas for a topological manifold preserve the natural differential structure of $\mathbf{R}^n$ (that is, if they are diffeomorphisms), the differential structure transfers to the manifold and turns it into a differentiable manifold.

In general the structure on the manifold depends on the atlas, but sometimes different atlases give rise to the same structure. Such atlases are called **compatible**.

7.6 CONSTRUCTION

A single manifold can be constructed in different ways, each stressing a different aspect of the manifold, thereby leading to a slightly different viewpoint.

7.6.1 Charts



The chart maps the part of the sphere with positive z coordinate to a disc.

Perhaps the simplest way to construct a manifold is the one used in the example above of the circle. First, a subset of $\mathbf{R}^2$ is identified, and then an atlas covering this subset is constructed. The concept of *manifold* grew historically from constructions like this. Here is another example, applying this method to the construction of a sphere:

Sphere with charts

A sphere can be treated in almost the same way as the circle. In mathematics a sphere is just the surface (not the solid interior), which can be defined as a subset of $\mathbf{R}^3$:

$$S = \{(x, y, z) \in \mathbf{R}^3 | x^2 + y^2 + z^2 = 1\}.$$

The sphere is two-dimensional, so each chart will map part of the sphere to an open subset of $\mathbf{R}^2$. Consider the northern hemisphere, which is the part with positive *z* coordinate (coloured red in the picture on the right). The function χ defined by

$$\chi(x,y,z) = (x,y),$$

maps the northern hemisphere to the open unit disc by projecting it on the (x, y) plane. A similar chart exists for the southern hemisphere. Together with two charts projecting on the (x, z) plane and two charts projecting on the (y, z) plane, an atlas of six charts is obtained which covers the entire sphere.

This can be easily generalized to higher-dimensional spheres.

7.6.2 Patchwork

A manifold can be constructed by gluing together pieces in a consistent manner, making them into overlapping charts. This construction is possible for any manifold and hence it is often used as a characterisation, especially for differentiable and Riemannian manifolds. It focuses on an atlas, as the patches naturally provide charts, and since there is no exterior space involved it leads to an intrinsic view of the manifold.

The manifold is constructed by specifying an atlas, which is itself defined by transition maps. A point of the manifold is therefore an equivalence class of points which are mapped to each other by transition maps. Charts map equivalence classes to points of a single patch. There are usually strong demands on the consistency of the transition maps. For topological manifolds they are required to be homeomorphisms; if they are also diffeomorphisms, the resulting manifold is a differentiable manifold.

This can be illustrated with the transition map $t = {}^{1}\!/\!_{s}$ from the second half of the circle example. Start with two copies of the line. Use the coordinate *s* for the first copy, and *t* for the second copy. Now, glue both copies together by identifying the point *t* on the second copy with the point ${}^{1}\!/\!_{s}$ on the first copy (the point $t = 0$ is not identified with any point on the first copy). This gives a circle.

Intrinsic and extrinsic view

The first construction and this construction are very similar, but they represent rather different points of view. In the first construction, the manifold is seen as embedded in some Euclidean space. This is the *extrinsic view*. When a manifold is viewed in this way, it is easy to use intuition from Euclidean spaces to define additional structure. For example, in a Euclidean space it is always clear whether a vector at some point is tangential or normal to some surface through that point.

The patchwork construction does not use any embedding, but simply views the manifold as a topological space by itself. This abstract point of view is called the *intrinsic view*. It can make it harder to imagine what a tangent vector might be.

n-Sphere as a patchwork

The *n*-sphere $\mathbf{S}^n$ is a generalisation of the idea of a circle (1-sphere) and sphere (2-sphere) to higher dimensions. An *n*-sphere $\mathbf{S}^n$ can be constructed by gluing together two copies of $\mathbf{R}^n$. The transition map between them is defined as

$$\mathbf{R}^n \setminus \{0\} \to \mathbf{R}^n \setminus \{0\} : x \mapsto x/\|x\|^2.$$

This function is its own inverse and thus can be used in both directions. As the transition map is a smooth function, this atlas defines a smooth manifold. In the case $n = 1$, the example simplifies to the circle example given earlier.

7.6.3 Identifying points of a Manifold

It is possible to define different points of a manifold to be same. This can be visualized as gluing these points together in a single point, forming a quotient space. There is, however, no reason to expect such quotient spaces to be manifolds. Among the possible quotient spaces that are not necessarily manifolds, orbifolds and CW complexes are considered to be relatively well-behaved.

One method of identifying points (gluing them together) is through a right (or left) action of a group, which acts on the manifold. Two points are identified if one is moved onto the other by some group element. If M is the manifold and G is the group, the resulting quotient space is denoted by M / G (or $G \backslash M$).

Manifolds which can be constructed by identifying points include tori and real projective spaces (starting with a plane and a sphere, respectively).

7.6.4 Cartesian Products

The Cartesian product of manifolds is also a manifold. Not every manifold can be written as a product.

The dimension of the product manifold is the sum of the dimensions of its factors. Its topology is the product topology, and a Cartesian product of charts is a chart for the product manifold. Thus, an atlas for the product manifold can be constructed using atlases for its factors. If these atlases define a differential structure on the factors, the corresponding atlas defines a differential structure on the product manifold. The same is true for any other structure defined on the factors. If one of the factors has a boundary, the product manifold also has a boundary. Cartesian products may be used to construct tori and finite cylinders, for example, as $\mathbf{S}^1 \times \mathbf{S}^1$ and $\mathbf{S}^1 \times [0, 1]$, respectively.

7.6.5 Manifold with Boundary

A *manifold with boundary* is a manifold with an edge. For example a sheet of paper with rounded corners is a 2-manifold with a 1-dimensional boundary. The edge of an n-manifold is an (n-1)-manifold. A disk (circle plus interior) is a 2-manifold with boundary. Its boundary is a circle, a 1-

manifold. A ball (sphere plus interior) is a 3-manifold with boundary. Its boundary is a sphere, a 2-manifold. (See also Boundary (topology)).

In technical language, a manifold with boundary is a space containing both interior points and boundary points.Every interior point has a neighbourhood homeomorphic to the open *n*-ball $\{(x_1, x_2, \ldots, x_n) \mid \Sigma x_i^2 < 1\}$. Every boundary point has a neighbourhood homeomorphic to the "half" *n*-ball $\{(x_1, x_2, \ldots, x_n) \mid \Sigma x_i^2 < 1 \text{ and } x_1 \geq 0\}$. The homeomorphism must send the boundary point to a point with $x_1 = 0$.

7.6.6 Gluing along boundaries

Two manifolds with boundaries can be glued together along a boundary. If this is done the right way, the result is also a manifold. Similarly, two boundaries of a single manifold can be glued together

Formally, the gluing is defined by a bijection between the two boundaries. Two points are identified when they are mapped onto each other. For a topological manifold this bijection should be a homeomorphism, otherwise the result will not be a topological manifold. Similarly for a differentiable manifold it has to be a diffeomorphism. For other manifolds other structures should be preserved.

A finite cylinder may be constructed as a manifold by starting with a strip **R** × [0, 1] and gluing a pair of opposite edges on the boundary by a suitable diffeomorphism. A projective plane may be obtained by gluing a sphere with a hole in it to a Möbius strip along their respective circular boundaries.

Check In Progress-II

Q. 1 Define chart.

Solution :

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

Q. 2 What is Manifold.

Solution :

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

7.7 CLASSES OF MANIFOLDS

7.7.1 Topological Manifolds

The simplest kind of manifold to define is the topological manifold, which looks locally like some "ordinary" Euclidean space $\mathbf{R}^n$. Formally, a topological manifold is a topological space locally homeomorphic to a Euclidean space. This means that every point has a neighbourhood for which there exists a homeomorphism (a bijective continuous function whose inverse is also continuous) mapping that neighbourhood to $\mathbf{R}^n$. These homeomorphisms are the charts of the manifold.

Usually additional technical assumptions on the topological space are made to exclude pathological cases. It is customary to require that the space be Hausdorff and second countable.

The *dimension* of the manifold at a certain point is the dimension of the Euclidean space charts at that point map to (number n in the definition). All points in a connected manifold have the same dimension. Some authors require that all charts of a topological manifold map to the same Euclidean space. In that case every topological manifold has a topological invariant, its dimension. Other authors allow disjoint unions of topological manifolds with differing dimensions to be called manifolds.

For most applications a special kind of topological manifold, a **differentiable manifold**, is used. If the local charts on a manifold are compatible in a certain sense, one can define directions, tangent spaces, and differentiable functions on that manifold. In particular it is possible to use calculus on a differentiable manifold. Each point of an *n*-dimensional differentiable manifold has a tangent space. This is an *n*-dimensional Euclidean space consisting of the tangent vectors of the curves through the point.

Two important classes of differentiable manifolds are **smooth** and **analytic manifolds**. For smooth manifolds the transition maps are smooth, that is infinitely differentiable. Analytic manifolds are smooth manifolds with the additional condition that the transition maps are analytic (a technical definition which loosely means that Taylor's theorem holds). The sphere can be given analytic structure, as can most familiar curves and surfaces.

A rectifiable set generalizes the idea of a piecewise smooth or rectifiable curve to higher dimensions; however, rectifiable sets are not in general manifolds.

7.7.2 Riemannian manifolds

To measure distances and angles on manifolds, the manifold must be Riemannian. A **Riemannian manifold** is an analytic manifold in which each tangent space is equipped with an inner product $\langle \, , \rangle$ in a manner which varies smoothly from point to point. Given two tangent vectors **u** and **v** the inner product $\langle \mathbf{u},\mathbf{v} \rangle$ gives a real number. The dot (or scalar) product is a typical example of an inner product. This allows one to define various notions such as length, angles, areas (or volumes), curvature, gradients of functions and divergence of vector fields.

Most familiar curves and surfaces, including *n*-spheres and Euclidean space, can be given the structure of a Riemannian manifold.

7.7.3 Finsler Manifolds

A **Finsler manifold** allows the definition of distance, but not of angle; it is an analytic manifold in which each tangent space is equipped with a norm, ||·||, in a manner which varies smoothly from point to point. This norm can be extended to a metric, defining the length of a curve; but it cannot in general be used to define an inner product.

Any Riemannian manifold is a Finsler manifold.

7.7.4 Lie groups

Lie groups are a particularly important class of manifolds. They were named after Sophus Lie (last name pronounced *Lee*). As well as having an inner product they also have the structure of a topological group, allowing a notion of multiplication of points on the manifold. Any compact Lie group can be given a Riemannian manifold structure. The circle can be given the structure of a Lie group — the circle group. The group structure is then the multiplicative group of all complex numbers with modulus 1.

A Euclidean vector space with the group operation of vector addition is an example of a non-compact Lie group. Other examples of Lie groups include special groups of matrices, which are all subgroups of the general linear group, the group of n by n matrices with non-zero determinant. If the matrix entries are real numbers, this will be an n^2-dimensional disconnected manifold. The orthogonal groups, the symmetry groups of the sphere and hyperspheres, are $n(n-1)/2$ dimensional manifolds, where $n-1$ is the dimension of the sphere. Further examples can be found in the table of Lie groups.v

7.7.5 Other Types of Manifolds

A **complex manifold** is a manifold modeled on $\mathbf{C}^n$ with holomorphic transition functions on chart overlaps. These manifolds are the basic objects of study in complex geometry. A one-complex-dimensional manifold is called a Riemann surface. (Note that an n-dimensional complex manifold has dimension $2n$ as a differentiable manifold.)

Infinite dimensional manifolds: to allow for infinite dimensions, one may consider Banach manifolds which are locally homeomorphic to Banach spaces. Similarly, Fréchet manifolds are locally homeomorphic to Fréchet spaces.

A **symplectic manifold** is a kind of manifold which is used to represent the phase spaces in classical mechanics. They are endowed with a 2-form that defines the Poisson bracket. A closely related type of manifold is a contact manifold.

7.8 SUMMARY

We study in this unit Topological Manifold and Its properties. We study classes of Manifold. We study Enriched Circles. We study Homomorphism and its properties.

7.9 KEYWORD

MANIFOLD : A collection of points forming a certain kind of set, such as those of a topologically closed surface or an analogue of this in three or more dimensions.

FINSELER : A *Finsler* manifold is a differentiable manifold M together with a ... is a continuous nonnegative function

7.10 EXERCISE

1 (Baire Category Theorem). A Hausdorff space that is locally compact satisfies: A countable union of closed sets without interiors has no interior.

2 . The set of rationals Q ⊂ R forms a metrizable space that does not admit a complete metric nor is it locally compact.

3 (Whitney Embedding, Final Version). An m-dimensional manifold M admits a proper embedding into R 2m+1

4 Let F : M → N be an immersion that is an embedding when restricted to the embedded submanifold S ⊂ M, then F is an embedding on a neighborhood of S

5 Let M ⊂ R n be an embedded submanifold. Then some neighborhood of the normal bundle of M in R n is diffeomorphic to a neighborhood of M in R n

7.11 ANSWER FOR CHECK IN PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 2.2

Q 2 Check in Section 2

Check in Progress-II

Answer Q. 1 Check in Section 5.1

Q 2 Check in Section 4.1

7.12 SUGGESTION READING AND REFERENCES

1. *Hubbard, John H.; West, Beverly H. (1995).* Differential Equations: A Dynamical Systems Approach. Part II: Higher-Dimensional Systems. *Texts in Applied Mathematics. 18. Springer. p. 204.* ISBN 978-0-387-94377-0.
2. Jump up to:[a] [b] Croom 1989, p. 7
3. Richeson 2008, p. 63
4. Aleksandrov 1969, p. 204

5. Jump up to:[a] [b] [c] Richeson (2008)
6. Listing, Johann Benedict, "Vorstudien zur Topologie", Vandenhoeck und Ruprecht, Göttingen, p. 67, 1848
7. Tait, Peter Guthrie, "Johann Benedict Listing (obituary)", Nature 27, 1 February 1883, pp. 316–317
8. *Fréchet, Maurice (1906). Sur quelques points du calcul fonctionnel. PhD dissertation.* OCLC 8897542.
9. Hausdorff, Felix, "Grundzüge der Mengenlehre", Leipzig: Veit. In (Hausdorff Werke, II (2002), 91–576)
10. Croom 1989, p. 129
11. Munkres, James R. Topology. Vol. 2. Upper Saddle River: Prentice Hall, 2000.
12. Adams, Colin Conrad, and Robert David Franzosa. Introduction to topology: pure and applied. Pearson Prentice Hall, 2008.
13. Allen Hatcher, Algebraic topology. (2002) Cambridge University Press, xii+544 pp. ISBN 0-521-79160-X, 0-521-79540-0.